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Wigner–Weyl–Moyal Formalism on Algebraic Structures

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We first introduce the Wigner–Weyl–Moyal formalism for a theory whose phase space is an arbitrary Lie algebra. We also generalize to quantum Lie algebras and to supersymmetric theories. It turns out that the noncommutativity leads to a deformation of the classical phase space: instead of being a vector space, it becomes a manifold, the topology of which is given by the commutator relations. It is shown in fact that the classical phase space, for a semisimple Lie algebra, becomes a homogeneous symplectic manifold. The symplectic product is also deformed. We finally make some comments on how to generalise to C^*-algebras and other operator algebras, too.

1. INTRODUCTION

The very powerful Wigner–Weyl–Moyal (WWM) formalism (Grossman, 1976; Royer, 1977; Dahl, 1982; Amiet and Ciblis 1991; Niehto, 1991; Várilly and Gracia-Bondía, 1989; Li, 1994a,b) is a way to associate with each operator describing a state, observable, or transition a function on phase space. This function is known as the Weyl symbol, or the Weyl transform of the corresponding operator. In this way the wave function (or rather the density matrix) is associated with a pseudo-distribution function known as the Wigner function. This function, denote it by F, is the closest analogue of the classical phase-space distribution, which enters, for instance, in the Boltzmann equation. It can, however, be nonpositive, and is hence not a proper distribution function—in most cases the Heisenberg uncertainty relation forbids the existence of such a proper distribution function. As first pointed out by Moyal, the Weyl transform generates a deformation, on the phase space, of the classical Poisson brackets and of the usual commutative product

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$$(f(q, p), g(q, p)) \to f(q, p)g(q, p) = (fg)(q, p)$$

The deformed product is denoted by * and is called the twisted product. It is in general noncommutative. The deformation of the Poisson bracket is what is known as the Moyal bracket,

$$\begin{aligned}[f(q, p), g(q, p)]_M &= f(q, p) * g(q, p) - g(q, p) * f(q, p) \\ &= i\hbar\{f(q, p), g(q, p)\}_{\text{PB}} + O(\hbar^2)\end{aligned}$$

It is this method we want to extend to a phase space which is not just that of quantum mechanics, but can be an arbitrary (finite or infinite dimensional) Lie algebra or, as will be shown later, super-Lie algebra, a quantum-Lie algebra, or a C^*-algebra.[2]

We will first review the standard WWM approach to the quantum mechanical phase space, i.e., to the Lie algebra h_n of the Heisenberg group in n dimensions. This will be done in terms of certain translation operators. This formalism will then be carried over into a second-quantized formulation by introducing a new basis, namely that of creation and annihilation operators. This will at once show us how to extend the formalism in two directions: (1) to an arbitrary Lie algebra, and (2) to fermionic degrees of freedom. These can then be combined to give a WWM formalism for super-Lie algebras. The way we derive the standard WWM approach will show some connection with quantum groups, and hence we will also be commenting on how to extend this formalism even further, into the realm of quantum deformed Lie algebras—quantum-Lie algebras. Finally we will study general operator algebras, and we will show that our method can be generalized to C^*-algebras. We finish with some comments on further generalizations and applications.

2. THE WWM APPROACH TO THE STANDARD PHASE SPACE

The standard phase space of quantum mechanics is given by $2n$ generators $\hat{q}_i$, $\hat{p}_i$ satisfying (we only treat bosons for now; we will, however, return to fermions later)

[2] Some abuse of notation is used here. When we say that a quantum mechanical phase space is given by (or simply *is*) some Lie algebra, what we mean is that any quantum physical observable is some function of the generators of this algebra, hence the quantum phase space is really the universal enveloping algebra U of the Lie algebra in question. It is, however, straightforward to go from the Lie algebra to its universal enveloping algebra—the algebra of formal power series with elements from the Lie algebra. Furthermore, one could just as well consider the skew field P of fractions of U, $P = \{u^{-1}v \mid u, v \in U\}$. This would correspond to an algebra of formal Laurent series (i.e., functions possibly with singularities), and the corresponding classical phase space would then consists of meromorphic functions.

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij} \qquad \text{with} \quad i, j = 1, \ldots, n \tag{1}$$

in units where $\hbar = 1$.

We know that these commutation relations can only be represented faithfully in terms of operators on some Hilbert space, leading to the standard formulation of quantum theory. We are interested in a phase-space formulation which as closely as possible resembles that of classical statistical mechanics, and we thus need a correspondence between observables represented by operators on the Hilbert space $H = L^2(X)$ (X is the coordinate space, q-space, i.e., an n-dimensional vector space) and functions on a $2n$-dimensional symplectic space, phase space, i.e., we want a map, the *Weyl map*, $\hat{A} \mapsto A_W(q, p)$, where $\hat{A}$ is an operator on H and A_W is some function on the classical phase space. Quantization as a general formalism related to the introduction of such symbols for operators was, I believe, first extensively studied by Berezin (Berezin, 1975; Unterberger and Upmai, 1994). Following Grossmann (1976), Royer (1977), and Dahl (1982), we introduce operators

$$\Pi(u, v) = \exp(i(u \cdot \hat{p} - v \cdot \hat{q})) \tag{2}$$

These satisfy

$$\Pi(u, v)\Pi(u', v') = \Pi(u + u', v + v')Q(u, v; u', v') \tag{3}$$

where

$$Q(u, v; u', v') = e^{i(uv' - vu')/2} \tag{4}$$

is a C-number function. This shows then that $\Pi(u, v)$ constitutes a ray representation of the Euclidean group $\mathbb{R}^{2n}$, the group of translations in the Euclidean plane.[3] The symplectic form $uv' - vu'$ appears in this formula for Q, and in fact this is the only way in which it appears. The symplectic form is dictated by (or contained in) the algebraic relations defining the Heisenberg algebra. We will see that for more general Lie algebras, the defining relations (more precisely, the Cartan decomposition) dictates a symplectic structure.

One easily proves

$$\Pi(u, v)\hat{p}\Pi(u, v)^{-1} = \hat{p} - v \tag{5}$$

$$\Pi(u, v)\hat{q}\Pi(u, v)^{-1} = \hat{q} - u \tag{6}$$

which gives us a physical picture of what these operators do: they are transla-

[3] I use the following notation for the most important sets of numbers: $\mathbb{N}$ is the natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rationals, $\mathbb{R}$ the reals, $\mathbb{C}$ the complex numbers, and $\mathbb{H}$ the quaternions. A general field (or even division ring) will be denoted by $\mathbb{F}$, while $\mathbb{T}$ denotes the torus, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} \simeq S^1$. Commutators and anticommutators will be denoted by $[\cdot, \cdot]$, and $\{\cdot, \cdot\}$, while Moyal and Poisson brackets will be characterized by subscripts M and PB, respectively.

tions in phase space. It also shows us that u acts like a C-number version of the Q-number $\hat{q}$ and v as a C-number version of the Q-number $\hat{p}$; this shows that $\{(u, v)\}$ can be identified with the *classical* phase space. There are no restrictions imposed upon u, v; hence the classical phase space becomes simply $\mathbf{R}^{2n}$.

We can use the operator $\Pi(u, v)$ to construct our map $\hat{A} \mapsto A_W(u, v)$ as follows. To each operator describing an observable we associate a function given by

$$A_W(u, v) = \mathrm{Tr}(\Pi(u, v)\hat{A}) \tag{7}$$

This can be inverted to give

$$\hat{A} = \int A_W(u, v)\Pi(u, v)\, du\, dv \tag{8}$$

Actually, this map is only an isomorphism when $\hat{A}$ lies in the space $\mathcal{B}^2(H)$ of Hilbert–Schmidt operators. And we thus have an isomorphism between the space of Hilbert–Schmidt operators on $L^2(\mathbf{R}^n)$ and the function space $L^2(\mathbf{R}^n \times \mathbf{R}^n)$. The function corresponding to the density matrix ρ is known as the *Wigner function* (strictly speaking, this is only the symplectic Fourier transform of the proper Wigner function). For a pure state ψ we have $\rho = |\psi\rangle\langle\psi|$ and hence

$$F(u, v) = \mathrm{Tr}(\Pi(u, v)|\psi\rangle\langle\psi|) = \langle\psi|\Pi(u, v)|\psi\rangle \tag{9}$$

which gives a geometric interpretation of the Wigner function: it is the expectation value of a reflection operator (the symplectic Fourier transform of the translation operator Π is a reflection operator). This Wigner function is the closest quantum cousin of the classical distribution function $f(q, p)$; it is, however, in general nonpositive.

The *Weyl map* $\hat{A} \mapsto A_W$ generates an algebra structure on $L^2(\mathbf{R}_n \times \mathbf{R}_n)$ via

$$(\hat{A}\hat{B})_W \equiv A_W * B_W \tag{10}$$

This product is known as the *twisted product*; it is noncommutative, but associative; hence with this product $L^2(\mathbf{R}_n \times \mathbf{R}_n)$ becomes a non-Abelian Banach algebra (a Hilbert algebra even). One can show[4]

$$f * g = f(u, v) \exp\left(-\frac{1}{2} i\hbar \frac{\partial}{\partial v} \cdot \frac{\partial}{\partial u}\right) g(u, v) \tag{11}$$

where $\partial/\partial v$ is understood always to act on $f(u, v)$ and the other derivative always to act on g. We have reinserted $\hbar$ for clarity.

[4] A few papers in the mathematics literature deal with twisted products for some classical groups; see, e.g., Moreno (1986).

As the twisted product is noncommutative, we can introduce a kind of commutator, known as the *Moyal bracket*

$$[f(u, v), g(u, v)]_{\mathrm{M}} \equiv f * g - g * f \tag{12}$$

One easily sees that

$$([\hat{A}, \hat{B}])_{\mathrm{W}} = [A_{\mathrm{W}}, B_{\mathrm{W}}]_{\mathrm{M}} \tag{13}$$

Furthermore,

$$[f, g]_{\mathrm{M}} = 2if \sin(\tfrac{1}{2}\hbar\Delta)g \tag{14}$$

where we have introduced the *bi-differential operator*

$$f \Delta g \equiv \frac{\partial f}{\partial v} \cdot \frac{\partial g}{\partial u} - (u \leftrightarrow v) = \{f, g\}_{\mathrm{PB}} \tag{15}$$

which is the bi-differential operator defining the classical Poisson brackets, $\{\cdot, \cdot\}_{\mathrm{PB}}$. Hence

$$([\hat{A}, \hat{B}])_{\mathrm{W}} = [A_{\mathrm{W}}, B_{\mathrm{W}}]_{\mathrm{M}} = i\hbar\{A_{\mathrm{W}}, B_{\mathrm{W}}\}_{\mathrm{PB}} + O(\hbar^2) \tag{16}$$

Thus the Moyal bracket is a deformation of the classical Poisson bracket. Such deformations of classical Poisson structures have also been studied in their own right in the mathematics literature (Etingof and Kazhdan, 1995a,b). Also note that this relation clarifies the usual Heisenberg quantization rule

$$\{\cdot, \cdot\}_{\mathrm{PB}} \rightarrow \frac{1}{i\hbar}[\cdot, \cdot]$$

One should note that the Wigner function considered as a mapping $\mathcal{B}^2 \rightarrow L^2(\mathbf{R}^{2n})$ is not unique; one can modify the definition by the inclusion of an arbitrary function (Cohen, 1966; Springborg, 1983; Dahl, 1992; Davidović *et al.*, 1995). Each such function corresponds to a different prescription for the ordering of operator products. The Wigner function is, however, the simplest of these functions, and the only one for which we do not need a "dual" for going the other way $L^2(\mathbf{R}^{2n}) \rightarrow \mathcal{B}^2$. I refer to Cohen (1966) and Dahl (1991) for further details. Furthermore, one could just as wel use a translation operator based on *all* the generators of the Lie algebra, i.e., using

$$\Pi_{\mathrm{alt}}(u, v, w) \equiv \exp(iu\hat{p} - iv\hat{q} + iw\hat{1})$$

and the classical "phase space" is now apparently $(2n + 1)$-dimensional (parametrized by u, v, w), but one should note that $\hat{1}$ lies in the center of the algebra (the Heisenberg algebra is a central extension of the algebra of translations $\mathbf{R}^{2n}$), hence including it simply amounts to multiplying the functions by a phase:

$$\Pi_{\text{alt}}(u, v, w) = e^{iw}\Pi(u, v)$$

and can thus be ignored. These comments will turn out to be useful when the generalization to arbitrary Lie algebras is attempted.

Fascinating as all this is, we nonetheless have to move on. We want to generalize the above-outlined beautiful formalism to the case where the phase space is not just the Heisenberg algebra h_n, but any Lie algebra **g**.

2.1. Creation and Annihilation Operators

We need one more step before we can safely generalize to arbitrary Lie algebras. All physical processes can be described in terms of creation and annihilation operators. For a simple (bosonic) quantum mechanical system we know that these are given in terms of the operators $\hat{q}$, $\hat{p}$ by

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{p} + i\hat{q}) \tag{17}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{p} - i\hat{q}) \tag{18}$$

i.e., by a simple rotation of the quantum phase space. We know that these operators satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{19}$$

$$[\hat{n}, \hat{a}] = -\hat{a} \tag{20}$$

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger \tag{21}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator.

We introduce a new family of operators

$$\tilde{\Pi}(\alpha, \beta) \equiv \exp(-i(\alpha \cdot \hat{a}^\dagger - \beta \cdot \hat{a})) \tag{22}$$

Then

$$\tilde{\Pi}(\alpha, \beta)\tilde{\Pi}(\alpha', \beta') = \tilde{\Pi}(\alpha + \alpha', \beta + \beta')\tilde{Q}(\alpha, \beta; \alpha', \beta') \tag{23}$$

where

$$\tilde{Q}(\alpha, \beta; \alpha', \beta') = \exp(\tfrac{1}{2}(\alpha\beta' - \beta\alpha')) \tag{24}$$

Thus we once again have the same structure as before—not surprisingly, the transformation $(q, p) \to (a^\dagger, a)$ is merely a rotation—but note the absence of the imaginary unit in $\tilde{Q}$; this is of course due to the absence of an i in the fundamental commutator relations in this basis.

The importance of this example is the following:

- Fermions can be described by a similar algebra, but with anticommutators; the quantities α, β then become Grassmann numbers. (This will be shown later.)
- We can treat fields by letting the operators carry a continuous index (an element in some vector space or manifold) and inserting delta functions where appropriate.
- Any Lie algebra, finite or infinite dimensional, can be written in a form with creation and annihilation operators together with "number operators" (a root decomposition).

We should proceed with caution here. The algebra now consists of $3n + 1$ generators, namely $\hat{a}$, $\hat{a}^\dagger$, $\hat{n}$, 1, and while 1 belongs to the center, and thus can be ignored, this is by now means the case for $\hat{n}$. Why not use

$$\P(\alpha, \beta, \gamma) \equiv \exp(-i\alpha \cdot \hat{a}^\dagger + i\beta \cdot \hat{a} - i\gamma \cdot \hat{n})$$

instead? This would clearly alter the relations:

$$\begin{aligned}\P(\alpha, \beta, \gamma)\P(\alpha', \beta', \gamma') = \exp(&-i(\alpha + \alpha') \cdot \hat{a}^\dagger + i(\beta + \beta') \cdot \hat{a} - i(\gamma + \gamma') \cdot \hat{n} \\ &\tfrac{1}{2}(\alpha \cdot \beta' - \alpha' \cdot \beta) + (\alpha \cdot \gamma' - \alpha' \cdot \gamma)\hat{a}^\dagger \\ &- (\beta \cdot \gamma' - \beta' \cdot \gamma)\hat{a} + \cdots)\end{aligned}$$

We note one thing: To any order the term involving the extra generator $\hat{n}$ looks like $i(\gamma + \gamma') \cdot \hat{n}$; there are no higher order terms. Nor does it alter the symplectic product. The new generator only modifies the expression for the deformed addition, i.e., the terms involving $\hat{a}$, $\hat{a}^\dagger$. The γ, γ' appear more or less as some arbitrary parameters. The problem can be traced back to the fact that $\hat{n}$ is *not* an independent quantity. Dependent quantities will be elements of the universal enveloping algebras, i.e., polynomials in the generators, and should thus not be included among the basic quantities—they should be nonlinear functions of the classical phase-space variables, and not independent coordinates. This distinction will become clearer as we consider semisimple Lie algebras in the sequel.

2.1.1. Some Comments: Quantum Planes and Fibers

We elaborate a little bit on the structure involved in the WWM formalism as outlined above. The essential quantity was seen to be the operator $\Pi(u, v)$. This then led to a deformation of the classical Poisson structure and to an isomorphism between the Hilbert–Schmidt operators and the functions on phase space. Now, this deformation can also come about in another way. Define

$$X = e^{\hat{q}}, \qquad Y = e^{\hat{p}} \tag{25}$$

Then

$$XY = qYX \tag{26}$$

where $q = \exp(i\hbar)$. Hence X, Y make up a noncommutative geometry, known as the *quantum plane* R_q^2 (Wess and Zumino, 1990), which is a deformation of the classical space R^2. The automorphism group of this quantum plane is then what is known as a *quantum group*, a deformed version of a classical Lie group.

Define now

$$X(u) = e^{u\hat{q}}, \qquad Y(u) = e^{u\hat{p}} \tag{27}$$

Then we have what we could call a *quantum fiber bundle* where the base space is R and the fiber at u is a copy of the quantum plane. The deformation parameter q develops a u dependence, so we have different deformations at different points (the fibers are of course still isomorphic, though). We further note the nonlocal "folding"

$$X(u)Y(v) = q(u, v)Y(v)X(u) \tag{28}$$

which holds even when $u \neq v$. Let us finally note that $\Pi(u, v)$ is essentially just $X(v)Y(u)$. These arguments then indicate that quantum groups will indeed appear upon quantization of classical theories. In fact, the entire formalism as presented here is very intimately related to the study of quantum groups [see, e.g., Etingof and Kazhdan (1995a,b) for a related study of deformations of Poisson–Lie algebras].

3. AN ARBITRARY LIE ALGEBRA

We now want to generalize the WWM approach to the case where the given quantum phase space is an arbitrary Lie algebra. Two special cases are particularly important, namely Abelian and semisimple algebras, and will be treated first. Then we will comment on how to generalize to non-Abelian, nonsemisimple Lie algebras.

3.1. Abelian Lie Algebras

For each natural number n there exists just one (up to isomorphism) Abelian Lie algebra $\mathbf{a}$ with $\dim \mathbf{a} = n$. And this Lie algebra is isomorphic to F^n, where F is the base field (e.g., the reals or the complex numbers). The universal enveloping algebra $U(\mathbf{a})$ can then be identified with the ring of formal power series in n (commuting) variables:

$$U(\mathbf{a}) = \mathsf{F}[[X_1, \ldots, X_n]] \tag{29}$$

Thus we simply take the vector space F^n to be our classical phase space $\Gamma^0_{\mathbf{a}}$,

$$\Gamma^0_{\mathbf{a}} \equiv \mathsf{F}^n \simeq \mathbf{a} \tag{30}$$

Note, however, that the name "phase space" is somewhat inappropriate, as $\Gamma^0_{\mathbf{a}}$ will in general not be a symplectic space—in fact it will only be so if n is even, in which case we have the canonical symplectic form

$$\omega_0(X, X') \equiv X \wedge X' \equiv \sum_{i=1}^{n/2} (X_i X_{i'+n/2} - X_{i+n/2} X'_i) \tag{31}$$

All the same, for simplicity we will stick to the name phase space even in the case where $n = \dim \mathbf{a}$ is odd.

We should notice that $\Gamma^0_{\mathbf{a}}$ is a flat manifold (it is a vector space). It will turn out that non-Abelian Lie algebras have nonflat phase spaces. In the Abelian case $C(\Gamma^0_{\mathbf{a}})$ is simply the space of all functions which have a formal Taylor expansion. In general, this will of course not be true.

As $\mathbf{a}$ is Abelian, so is $U(\mathbf{a})$ and hence so is $C(\Gamma^0_{\mathbf{a}})$, i.e., the twisted product is just the usual product of functions

$$f(X) * g(X) = f(X)g(X) \tag{32}$$

There is an analogy with the case of Abelian C^*-algebras here: the famous Gel'fand theorem (Bratteli and Robinson, 1979; Murphy, 1990) states that any Abelian C^*-algebra is isomorphic to either the space $C_0(X)$ of continuous functions vanishing at infinity or the space $C_b(X)$ of bounded functions on some locally compact Hausdorff space X. We will later come across suggestions that this relationship between the WWM formalism for Lie algebras as proposed here and the Gel'fand theory for C^*-algebras goes deeper than this. We can collect the above in the following definition.

Definition 1. Let $\mathbf{a}$ be an Abelian Lie algebra with $n = \dim \mathbf{a} < \infty$ over some field F; then:

1. The classical phase space becomes $\Gamma^0_{\mathbf{a}} \equiv \mathsf{F}^n \simeq \mathbf{a}$; when n is even this is a symplectic space.
2. $C(\Gamma^0_{\mathbf{a}}) \simeq \mathsf{F}[[X_1, \ldots, X_n]]$ is the set of all formal power series in n variables.
3. The twisted product on $C(\Gamma^0_{\mathbf{a}})$ becomes trivial $f * g = fg$.

3.2. Semisimple Lie Algebras

Many models in physics use not only the Heisenberg algebra, but also some finite- or infinite-dimensional Lie algebra $\mathbf{g}$. The obvious examples are

Yang–Mills theories, σ-models, current algebras, conformal field theory, and string theory. In a Yang–Mills theory the fields A_μ (and their conjugate momenta π^μ) are elements of some Lie algebra **g**; $A_\mu = A_\mu^k \lambda_k$, where $[\lambda_k, \lambda_l] = ic_{kl}{}^m \lambda_m$. The same goes for σ-models; in current algebras we have commutator relations between the various components of the currents, $[J_\mu^k(x), J_\nu^l(x')] = i\delta(x - x')\eta_{\mu\nu}c^{kl}{}_m J_\mu^m(x)$. In conformal field theory we have a family of fields $\phi_i(z, \bar{z})$ depending on two complex variables and satisfying the so-called *conformal bootstrap* (Fuchs, 1992)

$$\phi_i(z, \bar{z})\phi_j(w, \bar{w}) = d_{ij}^k(z, \bar{z}, w, \bar{w})\phi_k(w, \bar{w})$$

A similar situation arises in string theory. As we can see, this is more or less the generic situation in modern physics, and hence we need to extend our WWM formalism to phase spaces extending the Heisenberg algebra.

For clarity we will first develop the formalism for finite-dimensional semisimple Lie algebras, and then we will make the (rather straightforward) generalization to their loop algebras and (affine) Kac–Moody algebras.

From basic Lie algebra theory (see, e.g., Jacobson, 1962; Fuchs, 1992) we know that we can choose a convenient basis $\{E_\alpha, H^i\}$ for the semisimple Lie algebra **g** such that

$$[H^i, H^j] = 0 \tag{33}$$

$$[H^i, E_\alpha] = \alpha^i E_\alpha \tag{34}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta}, & \alpha + \beta \text{ a nonzero root} \\ \alpha_i H^i & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

where $N_{\alpha,\beta}$ are some constants. The elements H^i, $i = 1, \ldots, l$, span the Cartan subalgebra **h** of **g** and act as number operators. The remaining elements E_α act as creation and annihilation operators (depending on the sign of the root α). When α is a root, so is $-\alpha$, hence we can divide the elements E_α into pairs $E_{\pm\alpha}$. We thus suggest the following generalization (α positive):

$$a_i \mapsto E_{-\alpha}, \qquad a_i^\dagger \mapsto E_{+\alpha}, \qquad n_i \mapsto H^i \tag{36}$$

As our basic translation operator $\Pi(u, v)$ (u, v are now r-tuples, where dim $\mathbf{g} = n = 2r + l$, $l =$ rank $\mathbf{g} =$ dim **h**) we will thus use the following:

Definition 2. If **g** is a semisimple Lie algebra of finite dimension and $\mathbf{g} = \mathbf{g}_0 + \Sigma_{\alpha>0}(\mathbf{g}_\alpha + \mathbf{g}_{-\alpha})$ is a root decomposition with respect to the Cartan subalgebra $\mathbf{g}_0$, then we define the Weyl map in terms of

$$\Pi(u, \mathbf{v}) = \exp(iu^\alpha E_{+\alpha} - iv^\alpha E_{-\alpha} + i\lambda^j(u, v)H_j)$$

summing over positive roots.

In general we cannot *a priori* omit the Cartan element (it would in general not give rise to a bijective map), so we have to include them explicitly, but, on the other hand, they are the analogues of the number operators and should thus not be counted as "independent quantities, i.e., the parameters λ^i should not be independent coordinates, but instead $\lambda^i = \lambda^i(u, v)$. These dependent coordinates λ^i are related to an embedding of the phase space which is $(n - l = 2r)$-dimensional into an n-dimensional vector space.[5]

We cannot, however, simply take over the relation

$$\Pi(u, v)\Pi(u', v') = \Pi(u + u', v + v')Q(u, v; u', v')$$

Instead it will turn out that the vector sum $u + u'$, $v + v'$ gets deformed, as does the symplectic product $\xi \wedge \xi' = uv' - vu'$. Hence we can write ($\xi \equiv (u, v)$) locally

$$\Pi(\xi)\Pi(\xi') = \Pi(\xi \oplus \xi')Q(\xi \times \xi') \tag{37}$$

Here Q depends only upon central and Cartan elements (for $\mathbf{g}$ semisimple, and only upon elements in the maximal Abelian subalgebra otherwise, as will be explained later).

The extra noncommutativity of the phase space leads to a deformation of the vector-space structure of $\mathbb{R}^{2r}$, the deformed vector sum being $\oplus$. The explicit form for $\xi \oplus \xi'$ is found by using the Baker–Campbell–Hausdorff formula, but for simplicity we will wait until the example $\mathbf{g} = su_2$ below before we write it out explicitly. Note that this deformation of the vector space structure on $\mathbb{R}^{2r}$ implies that the classical phase space [(u, v) space] might not be a vector space, but just a manifold. We will denote it by Γ or $\Gamma_{\mathbf{g}}$ when we wish to emphasize which algebra it belongs to. The symplectic product $\wedge$ gets deformed to $\times$. The corresponding twisted product can be written in terms of a kernel Δ like

$$(f * g)(\xi) = \int_\Gamma \Delta(\xi, \xi', \xi'')f(\xi')g(\xi'')\, d\xi'\, d\xi'' \tag{38}$$

where

$$\begin{aligned}\Delta(\xi, \xi', \xi'') &= \mathrm{Tr}(\Pi(u, v)\Pi(u', v')\Pi(u'', v'')) \\ &= \mathrm{Tr}(e^{i\langle\xi\oplus\xi'\oplus\xi'', E\rangle}e^{i(\xi\times(\xi'\oplus\xi'')+\xi'\times\xi'', H)})\end{aligned} \tag{39}$$

where we have defined

[5] We should also be aware of the fact that using the matrix trace is perhaps not the most general procedure; instead one could define an abstract trace as a linear functional χ with the property $\chi(AB) = \chi(BA)$; as this implies $\chi([A, B]) = 0$, we see that the number of such possible generalizations can be labeled by elements of the first cohomology class $H^1(\mathbf{g})$ of the Lie algebra $\mathbf{g}$. There will in general be essentially two, namely $\chi(AB) = \mathrm{Tr}\, A\, \mathrm{Tr}\, B$ and $\chi(AB) = \mathrm{Tr}(AB)$. The first of these must be discarded, as it would imply $A_W \propto \mathrm{Tr}\, A$ for all A, which is clearly unsatisfactory; hence only the second alternative is usable.

$$\langle \xi, E \rangle = u^\alpha E_{+\alpha} - v^\alpha E_{-\alpha}$$
$$(x, H) = x_j H^j$$

In order to satisfy the same relations as for the Heisenberg algebra, we must demand that $\mathrm{Tr}(\Pi(\xi)\Pi(\xi')) \equiv K(\xi, \xi')$ is a reproducing kernel for $L^2(\Gamma)$. This is seen by inserting the definitions of A_W, B_W in

$$\int_\Gamma A_W(\xi) B_W(\xi)\, d\xi = \mathrm{Tr}(AB)$$

which allow us to express expectation values in terms of integrals over the classical phase space (let, for instance, $B = \rho$, the density matrix).

We have proven the following result:

Proposition 1. Let $\mathbf{g}$ be as in Definition 2 above; then:

1. $\dim \Gamma = \dim \mathbf{g} - \dim \mathbf{g}_0 = n$.
2. Writing $\xi = (u, v)$ (in a local coordinate patch), we have $\Pi(\xi)\Pi(\xi') = \Pi(\xi \oplus \xi')Q(\xi \times \xi')$ with Q only involving the Cartan elements.
3. The deformed addition is given by

$$\begin{pmatrix} u \\ v \\ \lambda(u, v) \end{pmatrix} \oplus \begin{pmatrix} u' \\ v' \\ \lambda(u', v') \end{pmatrix} = \begin{pmatrix} u + u' + \text{higher order terms} \\ v + v' + \text{higher order terms} \\ \lambda(u, v) + \lambda(u', v') \end{pmatrix}$$

whereas the deformed symplectic product is

$$\xi \times \xi' = \omega_0(u, v, u', v') + \text{higher order terms}$$

with

$$\omega_0(u, v, u', v') \equiv \sum_{\alpha > 0} (u_\alpha v'_\alpha - u'_\alpha v_\alpha)$$

Concerning the nature of $C(\Gamma)$ and products of Π with itself, we can say the following:

Proposition 2. Let Π be the "translation" operator defining the Weyl map; then the twisted product of two functions $f, g \in C(\Gamma)$ can be written in term of a kernel Δ

$$(f * g)(\xi) = \int f(\xi') g(\xi'') \Delta(\xi, \xi', \xi'')\, d\xi'\, d\xi''$$

where $d\xi$ is a measure invariant under the action of the corresponding Lie group. The kernel is given by

$$\Delta(\xi, \xi', \xi'') = \mathrm{Tr}\, \Pi(\xi)\Pi(\xi')\Pi(\xi'')$$

Furthermore, $K(\xi, \xi')$ given by

$$K(\xi, \xi') = \mathrm{Tr}\, \Pi(\xi)\Pi(\xi')$$

is a reproducing kernel for $L^2(\Gamma)$.

Proof. We have

$$f * g = \mathrm{Tr}\, \Pi(\Pi^{-1}(f)\Pi^{-1}(g))$$

where $\Pi^{-1}(f)$ denotes the inverse of the WWM map, i.e., formally,

$$\Pi^{-1}(f) = \int_\Gamma \Pi(\xi) f(\xi)\, d\xi$$

The formula for Δ now follows. Similarly, the expression for K and the requirement that it be a reproducing kernel follow from the condition

$$\mathrm{Tr}(AB) = \int_\Gamma A_{\mathrm{W}}(\xi) B_{\mathrm{W}}(\xi)\, d\xi$$

upon writing $A = \int \Pi A_{\mathrm{W}}\, d\xi$. QED

Before continuing with Kac–Moody alkgebras, let me comment on the suggested formalism and its relations with other authors' proposals. Several authors have studied the natural symplectic structure associated with a Lie algebra (see, for instance, Alekseev and Malkin, 1994); this symplectic structure is based on the *coadjoint orbit action*. Given a Lie group G, we construct the symplectic space $O_m = \{m' = \mathrm{Ad}^*(g)m \,|\, g \in G\}$, where m is some point. The symplectic structure is given by the *Kirilov–Kostant Poisson bracket*

$$\{f, g\}_{\mathrm{KKP}}(m) \equiv \langle m, [df(m), dh(m)]\rangle$$

Here $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathbf{g}$, the Lie algebra of G, and its dual $\mathbf{g}^*$. Kasperkovitz (1993) and Kasperkovitz and Peev (1994) have applied this symplectic structure to the WWM formalism. These proposals are relevant when the "coordinate manifold" is a Lie algebra and one then needs to find a phase space. For an arbitrary coordinate manifold M (i.e., q-space) the associated phase space is the cotangent bundle T^*M, so even if the global momentum space (p-space) is not defined, the phase space *is* well defined. It is this construction the coadjoint orbit formalism generalizes for M replaced by an arbitrary Lie algebra. But, *a priori*, systems do exist for which we can define neither a global coordinate manifold nor a global momentum space. Darboux' theorem (Woodhouse, 1992) asserts, though, that we can always define coordinates p, q locally satisfying the usual Poisson bracket relations.

The generalization of the WWM formalism proposed here is able to handle this situation easily, as it is based directly on the phase-space manifold and not on the coordinate manifold. What we in this paper are essentially doing is to reconstruct a topological space Γ by a ring of continuous functions $C(\Gamma)$ on it (i.e., essentially "pointless topology," or perhaps rather "pointless differential geometry").

Before the example, which will hopefully clarify the formalism somewhat, let me just briefly mention infinite-dimensional Lie algebras. Given a finite-dimensional Lie algebra **g** with generators λ^k, we can construct the corresponding infinite-dimensional Lie algebra of maps $S^1 \to \mathbf{g}$; this algebra is known as the *loop algebra* of **g**, and will be denoted by $\mathbf{g}_{\text{loop}}$. A basis for this Lie algebra is $\lambda^k_m = \lambda^k z^m$, where z is a complex number of modulus 1. The commutator relations are

$$[\lambda^k_m, \lambda^l_n] = ic^{kl}{}_j\lambda^h_{m+n} \tag{40}$$

This is probably the simplest way of generating infinite-dimensional Lie algebras. The more general class of *Kac–Moody algebras* (Kac, 1985; Fuchs, 1992) is based on a relaxation of the restraints on the Cartan matrix A^{ij}; interestingly this too leads to infinite-dimensional Lie algebras. An important subclass of these algebras, the so-called *affine Kac–Moody algebras* (defined by demanding the Cartan matrix to be positive semidefinite) can be viewed as a nontrivial central extension of a loop algebra, and a basis can be chosen such that

$$[H^i_m, H^j_n] = mG^{ij}\delta_{m+n,0}K \tag{41}$$

$$[H^i_m, E^\alpha_n] = \alpha^i E^\alpha_{m+n} \tag{42}$$

$$[E^\alpha_m, E^\beta_n] = N_{\alpha\beta}E^{\alpha+\beta}_{m+n} \tag{43}$$

$$[E^\alpha_m, E^{-\alpha}_{-m}] = \alpha_i H^i_m + mK \tag{44}$$

where α, β are roots, $N_{\alpha\beta} = 0$ if $\alpha + \beta$ is not a root, G^{ij} is some matrix, and K is the central generator. The eigenvalue of K is known as the *level*. Notice that the generators with $m = n = 0$ span a subalgebra, which is an ordinary Lie algebra. Affine Kac–Moody algebras can be included in our formalism by making the substitution $u_i \mapsto u^m_i$, $i = 1, 2, \ldots, r$; $m = 0, \pm 1, \pm 2, \ldots$; so each u_i gets replaced by an entire sequence leading to an infinite-dimensional classical phase space. In order to deal with nonaffine Kac–Moody algebras, we will have to go back to the general commutator relations, as no particular representation in terms of other algebras is known. If we just treat A^{ij} as an arbitrary matrix, we can include also these kinds of Kac–Moody algebras in our formalism—in principle at least.

4. AN EXAMPLE: $su_2 = so_3$

To really see the formalism at work, we will consider the simplest nontrivial example, namely $\mathbf{g} = su_2$. For simplicity we will work in the $s = 1/2$ representation only (later we will show that the result is independent of the choice of representation); the generators can then be chosen to be the Pauli matrices σ_i, from which we can define

$$\sigma_\pm = \frac{1}{\sqrt{2}}(\sigma_1 \pm i\sigma_2)$$

But it will be just as easy to work directly with σ_i instead and we will do this. The "translation" operator is then

$$\Pi(u, v) = \exp(iu\sigma_1 - iv\sigma_2 + i\lambda(u, v)\sigma_3) \tag{45}$$

which can be rewritten as (using the familiar properties of the Pauli matrices)

$$\Pi(u, v) = \cos\sqrt{u^2 + v^2 + \lambda^2} + i(u\sigma_1 - v\sigma_2 + \lambda\sigma_3)\frac{\sin\sqrt{u^2 + v^2 + \lambda^2}}{\sqrt{u^2 + v^2 + \lambda^2}} \tag{46}$$

The most important ingredient is the deformed addition and symplectic product. Defining $\xi = (u, v)$ and

$$\xi \wedge \xi' \equiv uv' - vu' \tag{47}$$

the usual h_n case would read

$$\Pi(\xi)\Pi(\xi') = \Pi(\xi + \xi')Q(\xi \wedge \xi')$$

with

$$Q(\xi \wedge \xi') = e^{i\xi\wedge\xi'}$$

This gets deformed to

$$\Pi(\xi)\Pi(\xi') = \Pi(\xi \oplus \xi')Q(\xi \times \xi') \tag{48}$$

where $\oplus$ is the deformed vector sum and $\times$ the deformed symplectic product

$$\xi \oplus \xi' = \xi + \xi' + \text{cubic terms} \tag{49}$$

$$\xi \times \xi' = \xi \wedge \xi' + \text{quartic terms} \tag{50}$$

Computing the first corrections, we get

$$\xi \oplus \xi' = \xi + \xi' + \tfrac{1}{3}(\xi \wedge \xi')(\xi' - \xi) + \text{higher order terms} \tag{51}$$

Now, it follows from the properties of the Pauli matrices

$$e^{i\sigma_j u} = \cos u + i\sigma_j \sin u$$

that the function Π can be expressed in terms of trigonometric functions, so we must demand periodicity in the arguments. This implies that the classical phase space Γ can be one of two spaces (up to diffeomorphism), namely the torus $S^1 \times S^1$ or the sphere S^2. It is the commutator relations which determine which of the two spaces we have. Our phase space cannot be written as a product space $U \times V$, where $u \in U$, $v \in V$, as $[\sigma_+, \sigma_-] = 2\sigma_3 \notin Z(\mathbf{g})$ [$Z(\mathbf{g})$ denotes the center of the Lie algebra] and hence the classical phase space must be S^2, as we would expect (Amiet and Ciblis, 1991; Niehto, 1991; Várilly and Gracia-Bondía, 1989). The torus would correspond to a Lie algebra

$$[E_+, E_-] = 0$$

$$[H, E_+] = aE_+$$

$$[H, E_-] = -bE_-$$

where a, b are arbitrary positive numbers. A more rigorous argument is given in the section on general properties.

The requirement $\mathrm{Tr}(AB) = \int_\Gamma A_W B_W \, d\xi$ together with $\mathrm{Tr}(A) < \infty$ for all A in the universal enveloping algebra of su_2 implies that $\|A_W\|_2^2 = \int_\Gamma |A_W|^2 \, d\xi < \infty$ for all $A_W \in C(\Gamma)$. Thus $C(\Gamma) \simeq L^2(S^2)$.

This shows that, although the classical phase space inherits an addition making it locally isomorphic to the vector space $\mathbf{R}^{2r}$, this isomorphism will in general only be local. Thus *the classical phase space will be some 2r-dimensional real, symplectic manifold.* The global topological structure of this manifold could (*a priori*) be representation dependent—we will return to this point later—but the example suggests that only the commutator relations matter. The essential point is[6]

$$\text{noncommutativity} \rightarrow \text{nonflatness}$$

Proposition 3. We can write the "translation" operator Π as

$$\Pi(u, v) = f_0(u, v) + \sigma \cdot f(u, v) \tag{52}$$

with

$$f_0(u, v) = \cos\sqrt{u^2 + v^2 + \lambda^2}$$

$$f_1(u, v) = iu \frac{\sin\sqrt{u^2 + v^2 + \lambda^2}}{\sqrt{u^2 + v^2 + \lambda^2}}$$

$$f_2(u, v) = -iv \frac{\sin\sqrt{u^2 + v^2 + \lambda^2}}{\sqrt{u^2 + v^2 + \lambda^2}}$$

[6] This actually only holds with some slight modifications, as will be explained later.

$$f_3(u, v) = i\lambda \frac{\sin\sqrt{u^2 + v^2 + \lambda^2}}{\sqrt{u^2 + v^2 + \lambda^2}}$$

Proof. Straightforward computation using the properties of the Pauli matrices. ■

The Weyl maps of the generators become

$$(1)_W = 2f_0(u, v) \tag{53}$$

$$(\sigma_i)_W = 2f_i(u, v) \tag{54}$$

The factors of two can be removed by multiplying the trace by $1/(2s + 1)$. We must demand $f_0 \equiv$ const, which is the same as requiring $u^2 + v^2 + \lambda^2 =$ const, i.e., we once again get $\Gamma \simeq S^2$. Normalizing such that $(1)_W = 1$, we get

$$u^2 + v^2 + \lambda^2 = \arccos^2 \tfrac{1}{2} \tag{55}$$

which then gives λ as a function of u, v.

We notice that, had we taken $\lambda = 0$, we would have arrived at the most unfortunate result $(\sigma_3)_W = 0$, i.e., we would map the non-Abelian algebra su_2 onto an Abelian one. Instead we have $\lambda = \pm(\text{const}^2 - u^2 - v^2)^{1/2} \neq 0$. We note that to lowest order the generators σ_1, σ_2 (or equivalently $\sigma_\pm$) get mapped to u, v, whereas $(\sigma_3)_W$ is quadratic, to lowest order, in (u, v). This is because the Cartan subalgebra of a semisimple Lie algebra can be obtained from the root spaces $\mathbf{g}_{\pm\alpha} = \mathbb{F}E_{\pm\alpha}$ as $[\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}] \subseteq \mathbf{g}^0 = \mathbf{h}$. The Cartan elements are in this way not truly independent quantities.

Proposition 4. The reproducing kernel $K(u, v; u', v')$ and the kernel of the twisted product Δ become

$$\frac{1}{2} K = 1 - f_j(u, v) f_k(u', v') \delta^{jk} \tag{56}$$

$$\frac{1}{2} \Delta = 1 - f_j(u, v) f_k(u', v') \delta^{jk} - f_j(u, v) f_k(u'', v'') \delta^{jk} - f_j(u', v') f_k(u'', v'') \delta^{jk}$$
$$+ \sum_{ijk} f_i(u, v) f_j(u', v') f_k(u'', v'') \tag{57}$$

Proof. Straightforward computation. ■

The proposed WWM formalism has a very beautiful representation in terms of well-known quantities. For the sake of generality we will work in a general irreducible representation corresponding to an angular momentum l. The translation operator can be expanded

$$\Pi(u, v) = \sum_{mm'} \Pi_{mm'}(u, v) | lm\rangle\langle lm' | \tag{58}$$

where

$$\begin{aligned}\Pi_{mm'}(u, v) &\equiv \langle lm' | \Pi(u, v) | lm\rangle = \langle lm' | e^{iu\sigma_1 - iv\sigma_2 + i\lambda\sigma_3} | lm\rangle \\ &\equiv D^l_{m'm}(R_{(u,v)})\end{aligned} \tag{59}$$

where $R_{(u,v)}$ is the rotation given by the angles u, v. The $D^l_{mm'}(R)$ is the usual representation matrix for rotations (Merzbacher, 1970). For $\mathbf{g} = h_n$, the Heisenberg algebra, $\Pi(\xi)$ constitutes a (ray) representation of the group of translations, whereas for $\mathbf{g} = su_2$ we get a (proper) representation of the group of rotations; the phase space becomes the orbits of these groups, i.e., the plane and the sphere, respectively. The Weyl map of an "operator" [i.e., a $(2l + 1) \times (2l + 1)$ matrix] A becomes

$$A_{\mathrm{W}}(u, v) = \sum_{mm'} D^l_{m'm}(R_{(u,v)})\langle lm | A | lm'\rangle \equiv \sum_{mm'} D^l_{m'm} A_{mm'} \tag{60}$$

A very beautiful result. At this point we should notice that our WWM formalism is slightly different from the "standard approach" developed by Várilly and Gracia-Bondía (1989). Our formulas are slightly simpler, as we do not have Clebsch–Gordon coefficients occurring explicitly. Their "translation" operator, which they denote by Δ^l, is essentially our translation operator Π; in fact $\Delta^{1/2} \sim Y_{00} + \Pi$.

The inverse Weyl map of a function is also interesting to compute. Let $f(u, v) = \sum_m f_m(u, v) | lm\rangle$ be some function in $C(S^2)$; then the corresponding operator, which we will denote by f^{W}, is simply

$$f^{\mathrm{W}} = \sum_{mm'} \int D^l_{m'm}(R_{(u,v)}) f_m(u, v)\, d\Omega\, | lm\rangle\langle lm' | \tag{61}$$

where $d\Omega$ denotes the measure on S^2.

We can also use the rotation matrices $D^l_{mm'}$ to write

$$\Delta(\xi, \xi', \xi'') = \mathrm{Tr}\, \Pi(\xi)\Pi(\xi')\Pi(\xi'') = \sum_m D^l_{mm}(R_\xi R_{\xi'} R_{\xi''}) \tag{62}$$

Let us finish this subsection by making a comment on the measure on Γ. Clearly this measure $d\mu$ has to satisfy a few requirements: (1) it must be a Borel measure (the σ-algebra must be given by the topology, such that continuous functions are measurable), (2) it must be a Radon measure, i.e., the measure of a bounded set is bounded, and finally (3) it must be invariant under the group G (i.e., Haar), of which the operators $\Pi(\xi)$ constitute a representation. For the Heisenberg algebra this implies that $d\mu$ is the usual Lebesgue measure, as this is the only translation-invariant Radon measure (up to a multiplicative constant), whereas for su_2 it implies that $d\mu = d\Omega$, the usual solid angle measure.

4.1. The Corresponding Loop and Kac–Moody Algebras

Let us also consider the corresponding loop algebra $(su_2)_{\rm loop}$, which will be our first example of an infinite-dimensional Lie algebra. The commutator relations are

$$[\sigma_j^n, \sigma_k^m] = 2i\varepsilon_{jk}{}^l\sigma_l^{n+m} \tag{63}$$

where $\sigma_j^n = \sigma_j z^n$ with $z \in S^1$. Let $\bar{\sigma}_j$ denote the sequence $\{\sigma_j^n\}_{n\in Z}$ and define $\bar{u} = \{u_n\}_{n\in Z}$. We then introduce our, by now familiar, translation operator:

Definition 3. For a loop algebra formed from a semisimple, finite-dimensional Lie algebra we set

$$\Pi_{\rm loop}(\bar{u}, \bar{v}) \equiv \exp(i(\bar{u}\cdot\bar{\sigma}_+ - \bar{v}\cdot\bar{\sigma}_- + \bar{\lambda}\cdot\bar{\sigma}_3)) \tag{64}$$

where

$$\sigma_\pm^n \equiv \frac{1}{\sqrt{2}}(\sigma_1^n \pm i\sigma_2^n) \tag{65}$$

with the obvious notation

$$\bar{u}\cdot\bar{\sigma}_j \equiv \sum_{n=-\infty}^{\infty} u_n\sigma_j^n$$

In terms of the basis $\{\sigma_\pm^n, \sigma_3^m\}$ the commutator relations are

$$[\sigma_+^n, \sigma_-^m] = 2\sigma_3^{n+m}, \qquad [\sigma_3^n, \sigma_\pm^m] = \pm\sigma_\pm^{n+m}$$

and we have

$$\Pi_{\rm loop}(\xi)\Pi_{\rm loop}(\xi') = \Pi_{\rm loop}(\xi \oplus \xi')Q_{\rm loop}(\xi \times \xi') \tag{66}$$

Now

$$\bar{u}\cdot\bar{\sigma}_j \equiv \sum_{n=-\infty}^{\infty} u_n\sigma_j^n = \left(\sum_{n=-\infty}^{\infty} u_n z^n\right)\sigma_j \equiv u(z)\sigma_j \tag{67}$$

so the translation operator for the loop algebra can be expressed in terms of that of the basic Lie algebra as[7]

$$\Pi_{\rm loop}(\xi) = \Pi(\xi(z)) \tag{68}$$

where $\xi(z) = \sum_n \xi_n z^n$ is an analytic function $S^1 \to \Gamma = S^2$. This is a general result. The classical phase space of the loop algebra is the space of functions

[7] Thus $\bar{u}$ must be restricted to series for which $\sum u_n z^n$ is a well-defined analytic function.

$S^1 \to \Gamma$, where Γ is the classical phase space belonging to the original Lie algebra. Symbolically

$$\Gamma(\mathbf{g}_{\text{loop}}) \equiv \Gamma(C^\infty(S^1 \to \mathbf{g})) \simeq C^\infty(S^1 \to \Gamma(\mathbf{g})) \tag{69}$$

The deformation function Q_{loop} can be expressed in terms of Q as

$$Q_{\text{loop}}(\xi \times \xi') = Q(\xi(z) \times \xi'(z)) \tag{70}$$

where

$$\xi(z) \times \xi'(z) \equiv \sum_{n,m=-\infty}^{\infty} (u_n v'_m - v'_n u'_m) z^{n+m} \tag{71}$$

Thus the generalization to the loop algebra of a given Lie algebra is trivial. The Kac–Moody algebra $\widehat{su_2}$ at level k can be obtained from the loop algebra as

$$[\sigma_3^n, \sigma_3^m] = km\delta_{n,-m}$$

$$[\sigma_3^m, \sigma_\pm^m] = \pm\sigma_\pm^{n+m}$$

$$[\sigma_+^n, \sigma_-^m] = 2\sigma_3^{n+m} + km$$

The translation operator is defined to be

$$\Pi_{\text{KM}}(\xi) = \Pi_{\text{loop}}(\xi) \tag{72}$$

but with this new nontrivial central extension it satisfies

$$\Pi_{\text{KM}}(\xi)\Pi_{\text{KM}}(\xi') = \Pi_{\text{KM}}(\xi \oplus \xi')Q_{\text{KM}}(\xi \times \xi') \tag{73}$$

The deformation function Q_{KM} differs from Q by terms proportional to k; its σ_3^n term is identical to that of the loop algebra, which means that Q_{KM} differs from Q by a C-number function:

$$Q_{\text{KM}}(\xi \times \xi') = \mathcal{Q}_k(\xi, \xi')Q(\xi(z) \times \xi'(z)) \tag{74}$$

Explicitly,

$$\mathcal{Q}_k(\xi, \xi') = 1 - k \sum_{n=-\infty}^{\infty} n(u_n v'_n - u'_n v_n) + O(k^2) \tag{75}$$

This is also a general result; for an arbitrary Lie algebra $\mathbf{g}$ each element u_n, v_n would be r-dimensional, $u_n = (u_n^1, \ldots, u_n^r)$, etc., and we have to include a sum over this extra index in the above formula, too, but otherwise the analysis holds.

We have now seen how the proposed formalism works for a simple example, $\mathbf{g} = su_2$. Furthermore, we have seen how to relate the WWM formalism for a loop algebra or a Kac–Moody algebra to that of the original algebra, by which these infinite-dimensional algebras are generated.

As a final comment, we should note that the relationship (72) implies that the two classical phase spaces $\Gamma_{\text{loop}}, \Gamma_{\text{KM}}$ will be identical; the correspondence rules (the Weyl maps) will be different, though, and, in the language of an earlier subsection, so would their corresponding quantum fiber bundles. We can summarize this in the following result:

Proposition 5. Let $\mathbf{g}$ be a finite-dimensional semisimple Lie algebra, and denote by $\mathbf{g}_{\text{loop}}$ and $\hat{\mathbf{g}}_k$ its corresponding loop and affine Kac–Moody algebra at level k, respectively. The corresponding classical phase spaces are denoted by $\Gamma_{\mathbf{g}}$, $\Gamma(\mathbf{g}_{\text{loop}})$, and $\Gamma(\hat{\mathbf{g}}_k)$, respectively, and their "translation" operators by Π, Π_{loop}, and Π_{KM}; then:

1. $\Gamma(\mathbf{g}_{\text{loop}}) \simeq C^{\infty}(S^1 \to \Gamma_{\mathbf{g}})$.
2. $\Pi_{\text{loop}}(\xi) = \Pi(\xi(z))$ and $Q_{\text{loop}}(\xi \times \xi') = Q(\xi(z) \times \xi'(z))$ with $z \in S^1$ and

$$\xi(z) \times \xi'(z) = \sum_{n,m=-\infty}^{\infty} (u_n v'_m - u'_n v_m) z^{n+m} + \text{higher order terms}$$

3. $\Gamma(\hat{\mathbf{g}}_k) \simeq \Gamma(\mathbf{g}_{\text{loop}})$.
4. $\Pi_{\text{KM}}(\xi) = \Pi_{\text{loop}}(\xi)$ and $Q_{\text{KM}}(\xi \times \xi') = \mathfrak{Q}_k(\xi, \xi') Q_{\text{loop}}(\xi \times \xi')$, where $\mathfrak{Q}_k$ depends on the level k as

$$\mathfrak{Q}_k(\xi, \xi') = 1 - k \sum_{n=-\infty}^{\infty} n(u_n v'_n - u'_n v_n) + O(k^2)$$

There is an immediate generalization of the loop algebras to the gauging of any finite-dimensional Lie algebra. The algebra of local gauge transformations is locally[8]

$$\tilde{\mathbf{g}}(M) = C^{\infty}(M \to \mathbf{g}) \tag{76}$$

from which

$$\Pi_{\tilde{\mathbf{g}}(M)}(u, v) = \Pi_{\mathbf{g}}(u(x), v(x)), \qquad x \in M \tag{77}$$

and we have the following result:

Corollary 1. With $\mathbf{g}$ a semisimple Lie algebra of finite dimension and M any manifold, we have

[8] The group is not given by this simple formula globally, since we do not take the principal bundle structure into account; globally, the correct group is the group preserving the corresponding principal bundle, see e.g. (Nash 1991). For simplicity, though, we will consider only this particular group, $C^{\infty}(\text{M}) \oplus \mathbf{g}$, also sometimes denoted by Map$(M, \mathbf{g})$.

$$\Gamma(C^{\infty}(M) \otimes \mathbf{g}) \simeq C^{\infty}(M \to \Gamma) = C^{\infty}(M) \otimes \Gamma(\mathbf{g}) \tag{78}$$

5. THE STRUCTURE OF THE CLASSICAL PHASE SPACE

Now, the classical phase space was constructed from a map $\Pi(u, v)$, and clearly it is closely related to the Lie groups with **g** as their Lie algebra. In fact, had λ been independent of (u, v), we would have gotten a local Lie group (Omishchik, 1993). Let G be the smallest connected Lie group with **g** as its Lie algebra (note that G might not be simply connected); this then acts transitively on Γ, and thus (Omishchik, 1993, 1994), $\Gamma \simeq G/H_0$, where H_0 is some subgroup. Hence the classical phase space is a homogeneous space. From the construction it follows that H_0 is essentially a Lie group with **h**, the Cartan subalgebra, as its Lie algebra; it is not, however, identical to simply exp(**h**), as we have to subtract the center. Hence, $H_0 = H\backslash Z$, where H is the smallest connected Lie group with **h** as its Lie algebra. Very often we have only a trivial center, so often $H_0 = H$. For $\mathbf{g} = su_2 = so_3$, we thus have $G = SO_3$ and $H = SO_2$, whereby (trivial center)

$$\Gamma_{su_2} \simeq \Gamma_{so_3} \simeq SO_3/SO_2 \simeq SU_2/U_1 \simeq S^2$$

as we saw earlier.

We notice that for **g** semisimple, **h**, and thus also H, will be Abelian, whereas for a more general Lie algebra it will just be nilpotent. We can consider H as the subgroup spanned by the diagonal matrices, when G is a matrix group. The case of semisimple Lie algebras simplifies enormously by the Abelianness of the Cartan group, since any Abelian Lie group has the form $\mathsf{F}^n \times \mathsf{T}^m$, where F is te base field and T is the torus ($\mathsf{T} = S^1$, i.e., essentially SO_2 or U_1). Hence for compact Lie groups $H = \mathsf{T}^l$.

We should furthermore notice that a homogeneous space is symplectic if it is of the form G/H_ω, where H_ω is the connected component of the kernel of some antisymmetric two-form ω (Gamkrelidze, 1991). An obvious such 2-form is

$$\omega_0(u, v, \lambda, u', v', \lambda') = \begin{pmatrix} u \\ v \end{pmatrix} \wedge \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{79}$$

where $\wedge$ is the canonical symplectic product on R^{2r}. Clearly $H = \text{Ker}\ \omega_0$. It is important to notice that this not a completely arbitrary choice. It happens to be the symplectic form suggested by the Cartan splitting of the Lie algebra, since this is precisely the symplectic form which appears to lowest order in $Q(u, v; u', v')$. As we have seen, ω_0 gets deformed to another antisymmetric 2-form ω, which can be found order by order from the Baker–Campbell–Hausdorff theorem. This new 2-form will again vanish on H and nowhere else, and hence Γ is indeed a symplectic manifold when $\mathbf{g}$ is semisimple. When $\mathbf{g}$ is *not* semisimple, however, the symplectic form will be degenerate. Thus we have the following result.

Proposition 6. For $\mathbf{g}$ a semisimple Lie algebra with $n = \dim \mathbf{g} < \infty$ with Cartan subalgebra $\mathbf{h}$ we have $\Gamma \simeq G/H$, where G, H are the smallest, connected Lie groups with $\mathbf{g}$, $\mathbf{h}$ as their Lie algebras. Furthermore, Γ is symplectic.

Now, this was based on the assumption that $\mathbf{g}$ was semisimple. For an arbitrary Lie algebra, this will not be the case. In general the Cartan subalgebra $\mathbf{h}$ is defined as a maximal nilpotent subalgebra which is its own normalizer, i.e.,

$$\underbrace{[\mathbf{h}, [\mathbf{h}, \ldots, [\mathbf{h}, \mathbf{h}] \ldots]]}_{n \text{ brackets}} = 0 \quad \text{(for a sufficiently large } n\text{)}$$

$$\{x \in \mathbf{g} \,|\, [x, \mathbf{h}] \subseteq \mathbf{h}\} = \mathbf{h}$$

For any representation $\rho: \mathbf{g} \to gl(V)$, where V is some vector space, we can then write (Omishchik, 1993, 1994)

$$V = \bigoplus_{i=1}^{r} V^{\lambda_i} \tag{82}$$

where

$$V^{\lambda} = \{v \in V \,|\, \exists m \in \mathbf{N} \forall x\colon (\rho(x) - \lambda(x))^m v = 0\} \tag{83}$$

The quantities λ are linearly independent functionals on $\mathbf{h}$, i.e., $\Phi_\rho = \{\lambda_1, \ldots, \lambda_r\} \subseteq \mathbf{h}^*$; they are the weights. A root is then defined as a nonzero weight in the adjoint representation, i.e., $\Delta = \Phi_{\text{ad}} \backslash \{0\}$. We still have a root decomposition

$$\mathbf{g} = \mathbf{h} \oplus (\oplus_{\alpha \in \Delta} \mathbf{g}_\alpha) \tag{84}$$

and

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] \begin{cases} \subseteq \mathbf{g}_{\alpha+\beta}, & \alpha + \beta \in \Phi \\ = 0, & \alpha + \beta \notin \Phi \end{cases} \tag{85}$$

$$B(\mathbf{g}_\alpha, \mathbf{g}_\beta) = 0, \qquad \alpha + \beta \neq 0 \tag{86}$$

where $B(\cdot, \cdot)$ is the Killing form. Hence we still have some degree of orthogonality of the different root spaces. Unfortunately, it no longer holds that $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$, so the roots no longer come in pairs. Thus the classical phase space, which we can still define as we *do* have a root decomposition, will no longer be even-dimensional, and *a fortiori* not symplectic, in the general case. Hence

$$\mathbf{g} \text{ semisimple} \Rightarrow \Gamma_{\mathbf{g}} \text{ symplectic}$$

The use of the Cartan algebra as suggested above would constitute one generalization to nonsemisimple algebras, but I would like to propose another one, which I think is more appropriate. The reason for the success of the formalism in the semisimple case can be traced back to the fact that for such algebras the maximal nilpoent and the maximal Abelian subalgebra coincide: that the Cartan algebra becomes Abelian. So it was actually the Abeliannes of **h** that was used. Furthermore, while Cartan algebras of semisimple Lie algebras are fairly unique (they are conjugate), this will not in general hold for Cartan subalgebras of general Lie algebras, whereas Abelian Lie algebras are characterized completely by the dimension and are thus unique (up to isomorphism). So what I propose to do is consider not a maximal nilpotent Lie subalgebra **h**, but a maximal Abelian subalgebra **a**. It will turn out that we will have to refine this a bit further, but for now let us just list the consequences.

Now, clearly Abelian Lie algebras are also nilpotent, so we can use the above decomposition (which actually only holds for complex Lie algebras and not in general for real ones) for any (real or complex or otherwise) Lie algebra **g**. The dimensionality $s = \dim \mathbf{a}$ will not, however, be equal to the rank l of the Lie algebra. Let us call this number the *Abelian rank*, written a-rank(**g**). Obviously

$$1 + \dim Z(\mathbf{g}) \leq \text{a} - \text{rank}(\mathbf{g}) \leq \text{rank}(\mathbf{g}) \tag{87}$$

Let Φ denote the set of weights λ_i in the adjoint representation, and let $\Delta = \Phi\backslash\{0\}$; then we once more have a decomposition

$$\mathbf{g} = \mathbf{g}_0 \oplus (\oplus_{\alpha\in\Delta}\mathbf{g}_\alpha) \tag{88}$$

with $\mathbf{a} = \mathbf{g}_0$ and

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] \begin{cases} = 0, & \alpha + \beta \notin \Phi \\ \subseteq \mathbf{g}_{\alpha+\beta}, & \alpha + \beta \in \Phi \end{cases} \tag{89}$$

$$[\mathbf{g}_0, \mathbf{g}_\alpha] \subseteq \mathbf{g}_\alpha \tag{90}$$

We should notice that this construction implies that two Lie algebras have the same classical phase space if and only if one is the central extension of the other or the one can be written as the direct sum of the other and an Abelian algebra. In other words, Abelian algebras get mapped to the singleton set $\{0\}$. This of course differs from the definition of Γ^0 for an Abelian algebra given earlier, but agrees with our calculations for su_2. In fact, this is the reason why we inserted the superscript 0 in the definition of the Abelian case. Furthermore, this implies that our formalism assigns the same classical phase space (up to isolated points, which can always be discarded on physical grounds) to two algebras $\mathbf{g}_1$, $\mathbf{g}_2$ ($\dim \mathbf{g}_2 \geq \dim \mathbf{g}_1$, say) which differ by the addition of an Abelian algebra $\mathbf{a}$ (i.e., $\mathbf{g}_2 = \mathbf{g}_1 + \mathbf{a}$) such that $[\mathbf{a}, \mathbf{g}_1] \subseteq Z(\mathbf{g}_2)$, for instance when $\mathbf{g}_2$ is a central extension of $\mathbf{g}_1$ or when the sum is direct. The only exception to this is when $\mathbf{g}_1$, say, is itself Abelian; then $\Gamma_{\mathbf{g}_2} \simeq \Gamma^0_{\mathbf{g}_1}$, so the formalism is consistent with our choice of phase space for an Abelian Lie algebra—an example is of course the Heisenberg algebra, which is a central extension of R^{2n}. Note, however, that even though the classical phase spaces coincide, their correspondence rules given by the operators $\Pi_{1,2}$, $Q_{1,2}$ differ, as will their quantum fiber bundles.

As will be shown in the next section, one can modify the definition of $\mathbf{a}$ to arrive at a scheme that works equally well for Abelian as for non-Abelian Lie algebras. This modification will not alter the results given thus far, though, results which we can summarize in the following proposition.

Proposition 7. Two finite-dimensional Lie algebras $\mathbf{g}_1$, $\mathbf{g}_2$ have the same classical phase spaces up to isolated points if and only if one is the semidirect sum of an Abeliam algebra $\mathbf{a}$ and the other, say $\mathbf{g}_2 = \mathbf{g}_1 + \mathbf{a}$, with $[\mathbf{a}, \mathbf{g}_1] \subseteq Z(\mathbf{g}_2)$. A special case is when $\mathbf{g}_2$ is a central extension of $\mathbf{g}_1$.

5.1. Nilpotent and Solvable Lie Algebras

Some particular important cases of nonsemisimple Lie algebras are the nilpotent and solvable algebras. Let us make a few comments on the WWM formalism of these. Recall that a Lie algebra $\mathbf{g}$ is solvable if its derived series, $(\mathbf{g}^{(i)})$, with $\mathbf{g}^{(i)} = [\mathbf{g}^{(i-1)}, \mathbf{g}^{(i-1)}]$ for $i \geq 1$ and $\mathbf{g}^{(0)} = \mathbf{g}$, becomes trivial after a certain number of steps, i.e., $\mathbf{g}^{(i)} = 0$ for some value of i. Similarly, a Lie algebra is nilpotent if the series $(\mathbf{g}_{(i)})$ with $\mathbf{g}_{(i)} = [\mathbf{g}, \mathbf{g}_{(i-1)}]$ becomes trivial after a certain number of steps. A nilpotent Lie algebra is also solvable, and any Lie algebra can be written as the semidirect sum of a solvable and a semisimple Lie algebra (Levi decomposition). Hence once we know how to deal with solvable algebras as we can in principle handle *any* Lie algebra.

As far as solvmanifolds (i.e., homogeneous spaces of a solvable Lie group) are concerned, let me just mention that both the Möbius band and the Klein bottle are both solvmanifolds, and that any solvmanifold can be written as a fiber bundle over a compact solvmanifold with fiber R^k for some k (Omishchik, 1993). When the manifold is even a nilmanifold (i.e., when G is nilpotent), then this fiber bundle can be trivialized. Indeed, if M is any nilmanifold, then (Omishchik, 1993)

$$M \simeq M^* \times \mathsf{R}^n \tag{91}$$

where M^* is a compact nilmanifold. If $M = G/H$, then $M^* = {}^aH/H$, where ${}^aH$ denote the *algebraic closure* (i.e., the closure in the Zariski topology) of H. Hence, when H comes from the maximal Abelian subalgebra of $\mathbf{g}$, the Lie algebra of G, then ${}^a\mathbf{h} = \mathbf{h}$, so ${}^aH/H$ is discrete, i.e.,

$$\Gamma \simeq \mathsf{R}^n \times \text{discrete group}, \qquad n = \dim \mathbf{g} - \dim \mathbf{h} \tag{92}$$

This makes the case of nilpotent Lie algebras very simple (as we already noticed when we dealt with the Heisenberg algebra).

One should notice that we can obtain solvable Lie algebras from nilpotent ones by the following exact sequence:

$$0 \to \mathbf{g}' \to \mathbf{g} \to \mathbf{g}/\mathbf{g}' \to 0 \tag{93}$$

When $\mathbf{g}$ is solvable, then $\mathbf{g}'$ is the nil-radical, i.e., the largest nilpotent subalgebra. Thus, solvable Lie algebras can be obtained as extensions of nilpotent Lie algebras by Abelian ones. We will return to extensions when we deal with C^*-algbras.

Now, *a priori* the suggested WWM map will not be a bijection for nonsemisimple Lie algebras, as we do not *a priori* have $\mathbf{g}_0 \subseteq \cup_{\alpha,\beta\in\Delta}[\mathbf{g}_\alpha, \mathbf{g}_\beta]$. Thus we need to refine our definition somewhat. For semisimple Lie algebras we have $\mathbf{g}' \equiv [\mathbf{g}, \mathbf{g}] = \mathbf{g}$; hence the maximal nilpotent or Abelian subalgebra of $\mathbf{g}$ is also the maximal nilpotent/Abelian subalgebra of the derived algebra $\mathbf{g}'$ and vice versa. What we need in the general case, then, is the maximal Abelian subalgebra, which has the largest overlap with the derived algebra. This is ensured if we pick $\mathbf{a}$ as the largest Abelian subalgebra of $\mathbf{g}'$.

Definition 4. Let $\mathbf{g}$ be a finite-dimensional Lie algebra and let $\mathbf{a}$ be the maximal Abelian subalgebra of $\mathbf{g}' = [\mathbf{g}, \mathbf{g}]$. With this, writing $\Delta = \Delta_+ \cup \Delta_-$, where Δ_+ consists of positive roots, Δ_- of negative roots, the "translation" operator becomes

$$\Pi(u, v) = \exp\left(i \sum_{\alpha\in\Delta_+} u_\alpha E_\alpha - i \sum_{\alpha\in\Delta_-} v_\alpha E_\alpha + i\lambda^j(u, v)H_j\right)$$

where the H_j generate $\mathbf{a}$.

Notice that for the two cases already treated, namely Abelian and semisimple Lie algebras, respectively, this definition agrees with the old results. When $\mathbf{g}$ is semisimple, then $\mathbf{g}' = \mathbf{g}$ and $\mathbf{a} = \mathbf{h}$ becomes the Cartan algebra, whereas for $\mathbf{g} = \mathsf{F}^d$ Abelian we have $\mathbf{g}' = 0$, leading to $\Gamma \simeq \mathbf{g}/\{0\} = \mathbf{g} = \mathsf{F}^d$. Furthermore, for $\mathbf{g} = h_1$, the Heisenberg algebra, we have $\mathbf{a} = \mathsf{R}1$, whence $\Gamma(h_1) \simeq h_1/\mathsf{R} \simeq \mathsf{R}^2$.

6. SOME FURTHER EXAMPLES

We saw that the classical phase space of $su_2 = so_3$ turned out to be S^2. Let us now consider a few more examples very briefly.

Let us start with the Lie algebra of the noncompact group $SU(1, 1)$; it consists of traceless 2×2 matrices (in the fundamental representation) which obey

$$XJ = -JX^\dagger, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \tag{94}$$

The commutator relations are

$$[H, X_1] = -2X_2 \tag{95}$$

$$[H, X_2] = -2X_1 \tag{96}$$

$$[X_1, X_2] = -2iH \tag{97}$$

and a representation is

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\sigma_2$$

$$H = iJ = i\sigma_3$$

We can get from a representation of su_2 to one of $su_{1,1}$ by making the transformation (a "Wick rotation")

$$\sigma_1 \mapsto \sigma_1 = X_1, \qquad \sigma_2 \mapsto -\sigma_2 = X_2, \qquad \sigma_3 \mapsto i\sigma_3 = H \tag{98}$$

Inserting this in $\Pi(u, v)$, we get

$$\Pi_{su_{1,1}}(u, v) = e^{iu\sigma_1 + iv\sigma_2 - \lambda\sigma_3} \tag{99}$$

For the su_2 case we could introduce spherical coordinates for (u, v, λ); here it turns out that we get the following coordinates:

$$u = z \cos \alpha \cosh \beta$$

$$v = z \sin \alpha \cosh \beta$$

$$\lambda = z \sinh \beta$$

allowing us to write

$$\Pi_{su_{1,1}}(u, v) = \cos z + i(\cos\alpha \cosh\beta\sigma_1 + \sin\alpha\cosh\beta\sigma_2 + i\sinh\beta\sigma_3)\sin z \tag{100}$$

and the classical phase space becomes

$$\Gamma(su_{1,1}) \simeq \{(u, v, \lambda) \in \mathbf{R}^3 \,|\, u^2 + v^2 - \lambda^2 = \text{const}\} \equiv S^{1,1} \tag{101}$$

i.e., a hyperboloid.

Now, from su_2 and $su_{1,1}$ we can construct a number of important Lie algebras, by noting (Omishchik, 1994) $so_4 = su_2 \oplus su_2$, $so_{2,2} = su_{1,1} \oplus su_{1,1}$ and $u_2^*(\mathbf{H}) = su_2 \oplus su_{1,1}$, where $\mathbf{H}$ denote the ring of quaternions. The Lie algebra $so_{3,1}$, the Lorentz algebra, can also be constructed by noting $so_{3,1} = sl_2(\mathbf{C})_{\mathbf{R}} = su_2 \oplus i \cdot su_2 = su_2 \otimes \mathbf{C}$, where $sl_2(\mathbf{C})_{\mathbf{R}}$ means $sl_2(\mathbf{C})$ considered as a real algebra. These Lie algebras consists of 4×4 matrices of the form

$$so_4 \simeq su_2 \oplus su_2 \qquad \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & a & b \\ -\beta & -a & 0 & c \\ -\gamma & -b & -c & 0 \end{pmatrix} \qquad (a, b, c), (\alpha, \beta, \gamma) \in \mathbf{R}^3$$

$$so_{3,1} \simeq sl_2(\mathbf{C})_{\mathbf{R}} \qquad \begin{pmatrix} 0 & i\alpha & i\beta & i\gamma \\ -i\alpha & 0 & a & b \\ -i\beta & -a & 0 & c \\ -i\gamma & -b & -c & 0 \end{pmatrix} \qquad (a, b, c), (\alpha, \beta, \gamma) \in \mathbf{R}^3$$

$$so_{2,2} \simeq su_{1,1} \oplus su_{1,1} \qquad \begin{pmatrix} 0 & x & i\alpha & i\beta \\ -x & 0 & i\gamma & i\delta \\ -i\alpha & i\gamma & 0 & z \\ -i\beta & -i\delta & -z & 0 \end{pmatrix} \qquad x, z \in \mathbf{R}, \alpha, \beta, \gamma, \delta \in \mathbf{R}$$

$$u_2^*(\mathbf{H}) \simeq su_2 \oplus su_{1,1} \qquad \begin{pmatrix} 0 & x & a & b \\ -x & 0 & \bar{b} & d \\ -a & -\bar{b} & 0 & \bar{x} \\ -b & -d & -\bar{x} & 0 \end{pmatrix} \qquad x, b \in \mathbf{C}, \quad a, d \in \mathbf{R}$$

We must thus find an expression for $\Gamma_{\mathbf{g}_1 \oplus \mathbf{g}_2}$. Let us start with $so_4 = su_2 \oplus su_2$.We simply get

$$\Pi_{so_4}(u_1, v_1, u_2, v_2) = \Pi_{su_2}(u_1, v_1)\Pi_{su_2}(u_2, v_2) \tag{102}$$

$$Q_{so_4}(u_1, v_1, u_2, v_2) = Q_{su_2}(u_1, v_1)Q_{su_2}(u_2, v_2) \tag{103}$$

This is a general result:

Proposition 8. If $\mathbf{g}_1$, $\mathbf{g}_2$ denote two Lie algebras, then

$$\Pi_{\mathbf{g}_1 \oplus \mathbf{g}_2} = \Pi_{\mathbf{g}_1} \Pi_{\mathbf{g}_2} \tag{104}$$

$$Q_{\mathbf{g}_1 \oplus \mathbf{g}_2} = Q_{\mathbf{g}_1} Q_{\mathbf{g}_2} \tag{105}$$

Similarly, if $\mathbf{g}$ can be written as the sum of two Lie algebras with $[\mathbf{g}_1, \mathbf{g}_2] \in Z(\mathbf{g})$, then

$$\Pi_{\mathbf{g}} = \Pi_{\mathbf{g}_1} \Pi_{\mathbf{g}_2}$$

$$Q_{\mathbf{g}} = Q_{\mathbf{g}_1} Q_{\mathbf{g}_2} q_Z$$

where q_Z is some element in $\exp(Z(\mathbf{g}))$. It also follows from this that

$$\Pi_{\mathbf{g}} = \Pi_{\mathbf{g}/\mathbf{h}} \Pi_{\mathbf{h}} \tag{106}$$

when $\mathbf{h}$ is any ideal in $\mathbf{g}$. Thus the classical phase spaces become

$$\Gamma_{\mathbf{g}_1 \oplus \mathbf{g}_2} = \Gamma_{\mathbf{g}_1} \times \Gamma_{\mathbf{g}_2} \tag{107}$$

$$\Gamma_{\mathbf{g}_1 + \mathbf{g}_2} = \Gamma_{\mathbf{g}_1} \times \Gamma_{\mathbf{g}_2} \qquad (\text{when} \quad [\mathbf{g}_1, \mathbf{g}_2] \in Z(\mathbf{g})) \tag{108}$$

We should emphasize once more that the classical phase spaces of an algebra and its central extensions are isomorphic (up to isolated points); the correspondence between algebra and functions on phase space is different, though, and hence so are the corresponding quantum fiber bundles. Such central extensions are of great importance when $\mathbf{g}_1 = \mathbf{g}_2$ the algebra $\mathbf{g}$ is then a *Heisenberg double* of $\mathbf{g}_1$.[9] In a typical gauge theory, for instance, we have two set of operators ϕ_k, π_k both of which span some Lie algebra $\mathbf{g}_1$ at each point x and each instant t. The algebra is not just the gauging of $\mathbf{g}_1 \oplus \mathbf{g}_1$, but a central extension of it, as we have to impose $[\phi_k(x, t), \pi_j(x', t')]_{t=t'} = i\delta(x - x')\delta_{jk}$, the canonical relation.

For the algebras just mentioned we have at once

$$\Gamma_{so_4} = S^2 \times S^2 \tag{109}$$

$$\Gamma_{so_{2,2}} = S^{1,1} \times S^{1,1} \tag{110}$$

$$\Gamma_{u_2^*(\mathrm{H})} = S^2 \times S^{1,1} \tag{111}$$

The Lorentz algebra is somewhat more complicated. It arises as a complexification of su_2, and there is thus a nontrivial automorphism exchanging the real and complex parts of a Lie element. This means that

[9] In fact, for $\mathbf{g}_1 = \mathbf{g}_2 = \mathrm{R}$ we get the usual Heisenberg algebra. This shows that the new correspondence which the central extension introduces can be seen as related to quantization.

$$\Gamma_{so_{3,1}} = \frac{SO_3 \times SO_3}{SO_2 \times SO_2} \tag{112}$$

where $SO_2 \times SO_2$ is embedded in some nontrivial way in $SO_3 \times SO_3$ because of this automorphism. But noting that $SO_{3,1}$ is thus a complexification of su_2, i.e., $so_{3,1} = su_2 \otimes \mathbf{C}$, we get

$$\Gamma_{so_{3,1}} = \Gamma_{su_2 \otimes \mathbf{C}} \simeq \Gamma_{su_2} \otimes \mathbf{C} = S^2 \otimes \mathbf{C} \tag{113}$$

i.e., we can view the phase space of a complexification as a kind of "complexification" of the original phase space.

Let us now move on to a Lie algebra of rank two, namely su_3, represented by the Gell-Mann matrices λ_i, $i = 1, \ldots, 8$. We would expect the classical phase space to have a dimensionality of $8 - 2 = 6$. The key ingredient in the su_2 case was the useful relation $\sigma_i\sigma_j = i\varepsilon_{ij}{}^k\sigma_k$, which allowed us to get a nice expression for $\Pi(u, v)$ in terms of trigonometric functions. For su_3 we can use

$$[\lambda_a, \lambda_b] = if_{ab}{}^c\lambda_c \tag{114}$$

$$\{\lambda_a, \lambda_b\} = \tfrac{4}{3}\delta_{ab} + 2d_{bb}{}^c\lambda_c \tag{115}$$

where f_{abc} is totally antisymmetric, whereas d_{abc} is totally symmetric. From this it follows that

$$\lambda_a\lambda_b = if_{ab}{}^c\lambda_c + \tfrac{2}{3}\delta_{ab} + d_{ab}{}^c\lambda_c \tag{116}$$

Thus any function f of the generators can be written as

$$f(\lambda) = f_0 + \lambda_a f^a \tag{117}$$

where f_0, f^a are complex numbers, independent of the generators. These can be obtained from f by taking traces:

$$f_0 = \tfrac{1}{3}\mathrm{Tr}\, f(\lambda)$$
$$f^a = \tfrac{1}{3}\mathrm{Tr}(f(\lambda)\lambda^a)$$

Particularly useful for us are monomials $(u \cdot \lambda)^n$; we write

$$(u \cdot \lambda)^n = a_n(u) + \lambda_a b_n^a(u) \tag{118}$$

the coefficients satisfying

$$a_{n+1} = \tfrac{2}{3}u \cdot b_n \tag{119}$$

$$b_{n+1}^a = a_n u^a + u^b b_n^c d^a{}_{bc} \tag{120}$$

with $a_0 = 1$, $a_1 = 0$, $b_0^a = 0$, $b_1^a = u^a$. Explicitly, the kernel Δ and the translation operator Π becomes

$$\Pi(u) = c_0(u) + \lambda^a c_a(u) \tag{121}$$

$$\begin{aligned}\Delta(u, v, w) = {} & c_0(u)c_0(v)c_0(w) + \tfrac{2}{3}\delta^{ab} \sum_{\text{perm}} c_a(u)c_b(v)c_0(w) \\ & + \tfrac{2}{3}(d_{abc} + if_{abc})c^a(u)c^b(v)c^c(w)\end{aligned} \tag{122}$$

where

$$c_0(u) \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} a_n(u)$$

$$c^a(u) \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} b_n^a(u)$$

The product of two translation operators becomes

$$\begin{aligned}\Pi(u)\Pi(v) = {} & c_0(u)c_0(v) + \tfrac{2}{3}\delta^{ab}c_a(u)c_b(v) \\ & + \lambda^c(c_0(u)c_c(v) + c_c(u)c_0(v) + (if^{ab}{}_c + d^{ab}{}_c)c_a(u)c_b(v))\end{aligned} \tag{123}$$

whereby the reproducing kernel, in this representation, reads

$$K(u, v) = c_0(u)c_0(v) + \tfrac{2}{3}\delta^{ab}c_a(u)c_b(v) \tag{124}$$

The classical phase space becomes

$$\Gamma_{su_3} = SU_3/S(U_1 \times U_1 \times U_1) = SU_3/U_1 \times U_1 \tag{125}$$

In general

$$\Gamma_{su_n} = SU_n/S(U_1^n) = SU_n/U_1^{n-1} \tag{126}$$

with $U_1^k = U_1 \times \cdots \times U_1$ (k factors). I do not think these homogeneous spaces have any name.

We can get some insight into the structure of Γ_{su_3} by evaluating the Weyl symbols of the generators. Using (116) and the fact that the generators are traceless, one easily sees (the factor of two can of course be removed by a suitable normalization of the trace)

$$(1)_w = 2c_0(u) \tag{127}$$

$$(\lambda_a)_{\mathrm{W}} = 2c_a(u) \tag{128}$$

thus we must once more demand $c_0 = \text{const}$, which imposes a constraint on the variable u^a, deforming the phase space from simply $\mathbf{R}^6$ to some 6-manifold, just like for su_2, where the requirement $f_0 = \text{const}$ implied $\Gamma_{su_2} \simeq S^2$.

Furthermore, the symbol of a product becomes

$$(\lambda_a \lambda_b)_W = 2c_0(u)\delta_{ab} + 2(if_{ab}{}^c + d_{ab}{}^c)c_c(u) \tag{129}$$

Comparing this with

$$(c_a * c_b)(u) = \int c_a(v)c_b(w)\Delta(u, v, w)\, dv\, dw \tag{130}$$

we get

$$\begin{aligned}\delta_{ab} &= \frac{3}{2}c_0^2 \int c_a(v)\, dv \int c_b(w)\, dw \\ &\quad + \delta^{cd} \int c_c(v)c_a(v)\, dv \int c_d(w)c_b(w)\, dw \end{aligned} \tag{131}$$

$$\begin{aligned}(d_{ab}{}^c + if_{ab}{}^c) &= \frac{1}{3}c_0 \int (c_c(v) + c_c(w))c_a(v)c_b(w)\, dv\, dw \\ &\quad + \frac{1}{3}(d_{a'b'}{}^c + if_{a'b'}{}^c) \int c^{b'}(v)c^{a'}(w)c_a(v)c_b(w)\, dv\, dw\, w \end{aligned} \tag{132}$$

which gives us some insight into the nature of the functions $c_a(u)$.

As an example of an infinite-dimensional Lie algebra we can consider the *Witt algebra*, i.e., the algebra of diffeomorphisms of the circle. The commutator relations are

$$[A_n, A_m] = (m - n)A_{m+n}, \qquad n, m \in \mathbb{Z} \tag{133}$$

Our largest Abelian subalgebra is the one generated by A_0; hence

$$\Pi(u, v) = \exp\left(i \sum_{n>0} (u_n A_n - v_n A_{-n}) + i\lambda A_0\right) \tag{134}$$

Now, from $A_n^\dagger = A_{-n}$ we see that $\bar{u}_n = v_n$; hence the classical phase space consists of sequences (u_n, v_n) of complex numbers such that $\bar{u}_n = v_n$ and $n = 1, 2, 3, \ldots$. These can be represented just as well by sequences (x_n) with $x_n \in \mathbb{C}$, $n \in \mathbb{Z}$ satisfying $x_0 = 0$ and $\bar{x}_n = x_{-n}$, which again can be interpreted as a Fourier series, i.e.,

$$\Gamma_{\text{Witt}} = \left\{ f \in C^\infty(S^1) \,|\, f(\theta) = \sum_{n=1}^{\infty} (c_n e^{in\theta} + c_{-n} e^{-in\theta} \right\} \tag{135}$$

$$= \{ f \in C^\infty(S^1) \,|\, f(0) = 0 \} \tag{136}$$

This contains also the spaces of L^p functions on S^1 which vanish at $\theta = 0$. The deformed sum is seen to be

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} \oplus \begin{pmatrix} u'_k \\ v'_k \end{pmatrix}$$
$$= \begin{pmatrix} u_k + u'_k + \frac{1}{2} i \sum_{n=1}^{k-1} (k-2n) u_k u'_{k-n} - \frac{1}{2} ik(u_k \lambda' - u'_k \lambda) \\ + \frac{1}{2} ik \sum_{n=1}^{k-1} (u_n v'_{n-k} - v_{n-k} u'_n) + \cdots \\ v_k + v'_k + \frac{1}{2} i \sum_{n=1}^{k-1} (k-2n) v_k v'_{n-k} + \frac{1}{2} ik(v_k \lambda' - v'_k \lambda) \\ - \frac{1}{2} ik \sum_{n=1}^{k-1} (v_n u'_{n-k} - u_{n-k} v'_n) + \cdots \end{pmatrix} \quad (137)$$

and the deformed symplectic product to be

$$\begin{pmatrix} u \\ v \end{pmatrix} \times \begin{pmatrix} u' \\ v' \end{pmatrix} = \sum_k k(v_k u'_k - u_k v'_k) + \cdots \quad (138)$$

similar to the results we found for the loop and Kac–Moody algebras.

Now, the *Virasoro algebra*

$$[L_n, L_m] = (m-n) L_{n+m} + \delta_{n,-m} c_n \quad (139)$$

is just a central extension of the Witt algebra and will hence have the same classical phase space. We have seen earlier that also the classical phase spaces of the loop algebras of semisimple Lie algebras and their corresponding Kac–Moody algebras could be interpreted as function spaces over the unit circle S^1. We will encounter more function spaces when we move on to consider C^*-algebras as well.

Let us also briefly consider a more odd Lie algebra. The simplest algebra in which $\mathbf{a} \cap \mathbf{g}' = 0$ is the two-dimensional solvable Lie algebra $[h, x] = x$; here the only weight is $\alpha = 1$. A simple representation is $h = x\, d/dx$, $x = x$. The dimensionality of the classical phase space is one, and from the noncommutativity we see that we can take $\Gamma \simeq S^1$. This algebra has been considered by Isham (1984) and Isham and Kakas (1984), in the context of developing a general quantization algorithm for nontrivial phase spaces. Some final important examples are the Poincaré algebra $iso(3, 1)$ and the Galilei algebra gal_3. The Poincaré algebra is the semidirect sum of $\mathbb{R}^4$ and $so(3, 1) = su_2 \otimes \mathbb{C}$. Clearly $\mathbb{R}^4$ is the maximal Abelian subalgebra, and we get a classical phase space of dimension $10 - 4 = 6$. In fact the space must essentially be $SU_2 \cdot SU_2 \simeq S^3 \cdot S^3$, where the dot denotes some kind of product. It is rather surprising that the dimensionality becomes six and not eight as one would have expected[10] and, furthermore, that it is a kind of product of two compact manifolds. For the Galilei algebra we get similarly a six-dimensional phase space (as in this case was to be expected), but this time $SU_2 \cdot \mathsf{R}^3 \simeq S^3 \cdot \mathsf{R}^3$, i.e., the limit $c \to \infty$ which leads from the Galilei algebra to the Poincaré algebra (c is the velocity of light), leads to an "unwrapping" of one S^3, or, equivalently, that the finiteness of the velocity of light leads to a compactification of R^3. This suggests that Inönü–Wigner contractions leads to a "decompactification" of the classical phase space.

We have succeeded in obtaining Lie algebras yielding a number of two-dimensional manifolds as their classical phase spaces as shown in Table I. We would like to suggest that any surface can be obtained in this way, and as an example we will construct a Lie algebra with the Möbius band as its classical phase space. The algebras in Table I exhaust all nontrivial three-dimensional Lie algebras, hence the dimensionality of the wanted Lie algebra must be at least four. Since the Möbius band is a solvmanifold, but not a nilmanifold, this algebra must be solvable, but not nilpotent. On the other hand, the cylinder and the Möbius band differ only in the latter being a nontrivial bundle, but otherwise they both have the same local structure $\mathsf{R} \times_{\text{loc}} S^1$, where the subscript on $\times_{\text{loc}}$ is there to remind us that the product is only local in general. So let us start with the algebra behind the cylinder

$$[h, e] = e, \qquad [h, f] = [e, f] = 0$$

and let us add a fourth generator g mixing e, f,

$$[g, e] = \alpha f, \qquad [g, f] = \beta e$$

The Jacobi identity then implies $\alpha = 0$. We furthermore find $\mathbf{g}' = \{e\}$, i.e., $\mathbf{g}'' = 0$, so the algebra is solvable, while $\mathbf{g}^n = \mathbf{g}'$, so the algebra is not nilpotent. The largest Abelian subalgebra is $\mathbf{h} = \text{span}\{h, g\}$, and hence the dimensionality of the classical phase space is indeed two. Since Γ is a

Table I. Some Particularly Simple Two-Dimensional Manifolds and Their Corresponding Lie Algebras

Space		Algebra	
Plane	R^2	$[e, f] = h\ [e, h] = [f, h] = 0$	h_1
Cylinder	$\mathsf{R} \times S^1$	$[e, f] = [h, f] = 0\ [h, e] = e$	
Torus	$S^1 \times S^1$	$[e, f] = 0\ [h, e] = ae\ [h, f] = -bf$	
Sphere	S^2	$[e, f] = h\ [h, e] = e\ [h, f] = -f$	$su_2 = so_3 = sl_2$
Hyperboloid	$S^{1,1}$	$[e, f] = -h\ [h, e] = e\ [h, f] = -f$	$su_{1,1} = so_{2,1} = sl_{1,1}$

[10] This might be due to the mass-shell constraint $p^2 = m^2$ for the four-momentum together with the requirement that the particle move along a timelike geodesic, though.

solvmanifold of dimension two, it has the form of a (nontrivial) fiber bundle with fiber $\mathbb{R}$ over some compact, one-dimensional manifold M_1,

$$\Gamma \simeq \mathbb{R} \times_{\text{loc}} M_1$$

and it is easy to see that the only possibility is $M_1 = S^2$, from which we get

$$\Gamma \simeq \text{Möbius band} \tag{140}$$

One could then go on to find Lie algebras corresponding to surfaces of genus more than one, and, furthermore, to relate the topological characteristics (Euler number, Stiefel–Whitney classes) to algebraic properties of the Lie algebras—a kind of generalized index theorem—a point I plan to return to in a subsequent paper.

7. FERMIONIC DEGREES OF FREEDOM

Fermions are described by *anticommuting* creation and annihilation operators

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \tag{141}$$

$$\{a_i, a_j^\dagger\} = \delta_{ij} \tag{142}$$

We have no classical phase space at our disposal. So we cannot construct an isomorphism between an algebra of operators and a Hilbert space of functions on some vector space (or manifold), i.e., as a space of functions with C-number arguments. Rather, we have to define Grassmann numbers (which we will also refer to as G-numbers), abstract quantities satisfying

$$\{\theta_i, \theta_j\} = \{\bar{\theta}_i, \bar{\theta}_j\} = \{\theta_i, \bar{\theta}_j\} = 0$$

We can treat these as "coordinates" and their corresponding differential operators ∂_i, $\bar{\partial}_i$ as the "momentum" variables.

The generalization is now straightforward.

Definition 5. For fermionic creation and annihilation operators a, $a^\dagger$ we put

$$\Pi(\theta, \eta) \equiv \exp(i\theta a^\dagger - i\eta a) \tag{143}$$

where θ, η are G-numbers anticommuting with the second-quantization operators as well.

This operator will be our basis for developing a WWM formalism for fermionic degrees of freedom. The following proposition is trivial:

Proposition 9. The "translation" operator satisfies

$$\Pi(\theta, \eta)\Pi(\theta', \eta') = \Pi(\theta + \theta', \eta + \eta')Q(\theta, \eta; \theta'\eta') \tag{144}$$

where

$$Q(\theta, \eta; \theta', \eta') = \exp(\theta\eta' + \eta\theta') \tag{145}$$

We notice that this is in fact a C-number, being the product of two G-numbers. We also note that the sign in this G-symplectic product differs from the symplectic product of two C-numbers. No deformation of the sum or the symplectic product occurs here, as the G-numbers are nilpotent, $\theta^2 = \eta^2 = 0$. The Wigner function which follows from this has been derived independently by Abe (1992).

We easily get

$$a_W = i\theta \tag{146}$$

$$(a^\dagger)_W = -i\eta \tag{147}$$

Thus the conjugation of functions becomes

$$(f(\theta, \eta))^* = \bar{f}(\eta, \theta) \tag{148}$$

where the bar denotes Grassmann conjugation and the twisted product becomes

$$\begin{aligned}(f * g)(\theta, \eta) = {} & 2(f_4 g_4 + 3f_3 g_2 - f_2 g_3 + 2f_1 g_4 + 2f_4 g_1) \\ & + 2(2f_1 g_2 - 2f_2 g_1 - 3f_2 g_4 - 3f_4 g_2)\theta \\ & + 2(2f_3 g_1 - 2f_1 g_3 - f_3 g_4 - f_4 g_3)\eta \\ & + 2(2f_4 g_4 - 6f_3 g_2 + 2f_2 g_3)\theta\eta \end{aligned} \tag{149}$$

where we have written $f = f_1 + f_2\theta + f_3\eta + f_4\theta\eta$ and similar for g. Contrasting this formula for the twisted product with the usual product

$$\begin{aligned}(fg)(\theta, \eta) = {} & f_1 g_1 + (f_1 g_2 + f_2 g_1)\theta + (f_1 g_3 + f_3 g_1)\eta \\ & + (f_1 g_4 + f_4 g_1 + f_2 g_3 - f_3 g_2)\theta\eta\end{aligned}$$

we see that the WWM formalism introduces even more noncommutativity. With fermionic degrees of freedom within reach, the extension to super-Lie algebras (DeWitt, 1984; Wess and Bagger, 1992) is straightforward.

7.1. Clifford and Spin Algebras

I do not know of any concrete examples where the quantum phase space is a Clifford algebra, except of course the already treated case of $\mathbf{g} = su_2$. Nevertheless it might be interesting to have a look at the WWM formalism for such algebras. Now, a Clifford algebra $C(r, s)$ is by definition an algebra in $n = r + s$ generators γ_a satisfying

$$\{\gamma_a, \gamma_b\} = 2g_{ab} \tag{150}$$

where g_{ab} is a metric with signature (r, s). We will simply assume

$$g_{ab} = \eta_{ab} \equiv \text{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}) \tag{151}$$

Note that the definition implies $(\gamma_a)^2 = \pm 1$, hence $\dim C(r, s) = 2^{r+s}$. The case of su_2 corresponds to $r = 2$, $s = 0$ with $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\sigma_3 = \frac{1}{2}\gamma_1\gamma_2$. Our "classical coordinates" ξ_i will be taken to be G-numbers anticommuting with the γ-matrices, $\{\xi_i, \gamma_j\} = 0$. This would give a new representation of a classical phase space of this algebra, in other words, su_2 as a Lie algebra must be treated differently from su_2 as a Clifford algebra.

Let me just sketch the results for the usual Clifford algebra $C(1, 3)$, the *Dirac algebra*. The translation operator is defined in the most natural way as follows:

Definition 6. Let Γ^I denote the generators of the Clifford algebra $C(r, s)$; then

$$\Pi(\xi) = e^{i\xi_I\Gamma^I} \tag{152}$$

where ξ_I are G-numbers anticommuting with the Clifford generators.

In the particularly important case of the Dirac algebra we have the following proposition:

Proposition 10. In the particular case of the Dirac algebra $C(1, 3)$, we have

$$\Pi(\xi) = (1 + i\xi_0)\, 1 + \tilde{\xi}_0\gamma^5 + \xi_m\gamma^m + \tilde{\xi}_m\gamma^5\gamma^m + \xi_{mn}\sigma^{mn}$$

where $\sigma_{mn} = \frac{1}{4}i[\gamma_m, \gamma_n]$ and where $m, n = 0, 1, 2, 3$.

Proof. For $r = 3$, $s = 1$—the Dirac algebra—we have

$$\Pi(\xi) \equiv \exp(i\xi_0 1 + i\tilde{\xi}_0\gamma_5 + i\xi_m\gamma^m + i\tilde{\xi}_m\gamma_5\gamma^m + i\xi_{mn}\sigma^{mn}) \tag{153}$$

It has the decomposition (as does any function on a Clifford algebra)

$$\Pi(\xi) = \Pi_0(\xi) + \tilde{\Pi}_0(\xi)\gamma^5 + \Pi_i(\xi)\gamma^i + \tilde{\Pi}_i(\xi)\gamma^i\gamma^5 + \Pi_{ij}(\xi)\sigma^{ij} \tag{154}$$

with

$$\Pi_0(\xi) \equiv \tfrac{1}{4}\text{Tr}\Pi(\xi) \qquad \text{(scalar)}$$

$$\tilde{\Pi}_0(\xi) \equiv \tfrac{1}{4}\text{Tr}(\Pi(\xi)\gamma^5) \qquad \text{(pseudoscalar)}$$

$$\Pi_i(\xi) \equiv \tfrac{1}{4}\mathrm{Tr}(\Pi(\xi)\gamma_i) \qquad \text{(vector)}$$
$$\tilde{\Pi}_i(\xi) \equiv \tfrac{1}{4}\mathrm{Tr}(\Pi(\xi)\gamma_i\gamma_5) \qquad \text{(axial vector)}$$
$$\Pi_{ij}(\xi) \equiv \tfrac{1}{4}\mathrm{Tr}(\Pi(\xi)\sigma_{ij}) \qquad \text{(tensor)}$$

But, as the coefficients are G-numbers, we have quite simply

$$\Pi_0(\xi) = 1 + i\xi_0$$
$$\tilde{\Pi}_0(\xi) = \tilde{\xi}_0$$
$$\Pi_m(\xi) = \xi_m$$
$$\tilde{\Pi}_m(\xi) = \tilde{\xi}_m$$
$$\Pi_{mn}(\xi) = \xi_{mn}$$

QED.

Thus, it also follows that

$$(\gamma_m)_W = \Pi_m(\xi) = i\xi_m \tag{155}$$

while

$$(1)_W = \Pi_0(\xi) = 1 + i\xi_0 \tag{156}$$

and so on. It follows from this that it is natural to demand $\xi_0 = 0$, which will lead to a dimensionality of Γ of dim $C(r, s) - 1 = 2^{r+s} - 1$. Thus we have the following result:

Proposition 11. Let Γ denote the classical phase space of a Clifford algebra $C(r, s)$; then

$$\dim \Gamma = \dim C(r, s) - 1 = 2^{r+s} - 1$$

as a Grassmann space.

One should note that this always gives an odd-dimensional space for any values of r, s. For the Clifford algebra su_2 we thus have an alternative classical phase space, namely a 3-dimensional Grassmann space.

By a direct computation one proves the following:

Proposition 12. For the case of the Dirac algebra $C(1, 3)$, the following results hold: The product of two "translation" operators is

$$\Sigma(\xi, \xi') \equiv \Pi(\xi)\Pi(\xi') = \Sigma_0 + \tilde{\Sigma}_0\gamma^5 + \Sigma_i\gamma^i + \tilde{\Sigma}_i\gamma^i\gamma^5 + \Sigma_{ij}\sigma^{ij} \tag{157}$$

where

$$\Sigma_0 = \tilde{\xi}_0\tilde{\xi}'_0 - 4i(\eta^{mp}\eta^{nq} - \eta^{mq}\eta^{np})\xi_{mn}\xi'_{pq} \tag{158}$$

$$\tilde{\Sigma}_0 = -4i\varepsilon^{mnpq}\xi_{mn}\xi'_{pq} \tag{159}$$

$$\begin{aligned}\Sigma_m = &-\tilde{\xi}_0\xi'_m + \tilde{\xi}_m\tilde{\xi}'_0\\ &+ 4i(\eta^{np}\delta^q_m - \eta^{nq}\delta^p_m)(\xi_n\xi'_{pq} + \xi_{pq}\xi'_n) - 4i\varepsilon^{npq}{}_m(\tilde{\xi}_n\xi'_{pq} + \xi_{pq}\tilde{\xi}'_n)\end{aligned} \tag{160}$$

$$\begin{aligned}\tilde{\Sigma}_m = &-\tilde{\xi}_0\xi'_m + \xi_m\tilde{\xi}'_0\\ &-4i\varepsilon^{npq}{}_m(\xi_n\xi'_{pq} - \xi_{pq}\xi'_n) + i(\eta^{np}\delta^q_m - \eta^{nq}\delta^p_m(\tilde{\xi}_n\xi'_{pq} + \xi_{pq}\tilde{\xi}'_n)\end{aligned} \tag{161}$$

$$\begin{aligned}\Sigma_{mn} = &\, 4i\varepsilon_{mn}{}^{pq}(\xi_{pq}\tilde{\xi}'_0 + \tilde{\xi}_0\xi'_{pq})\\ &+ 4i(\eta^{pq}\eta^{rs}\eta_{mn} - \eta^{qr}\delta^s_m\delta^p_n + \eta^{rs}\delta^p_m\delta^q_n - \eta^{sp}\delta^q_m\delta^r_n)\xi_{pq}\xi'_{rs}\end{aligned} \tag{162}$$

The reproducing kernel $K(\xi, \xi')$ becomes

$$\begin{aligned}K(\xi, \xi') &\equiv \tfrac{1}{4}\mathrm{Tr}(\Pi(\xi)\Pi(\xi')) = \tfrac{1}{4}\Sigma_0(\xi, \xi')\\ &= \tilde{\xi}_0\tilde{\xi}'_0 - 4i(\eta^{mp}\eta^{nq} - \eta^{mq}\eta^{np})\xi_{mn}\xi'_{pq}\end{aligned} \tag{163}$$

while the kernel for the twisted products takes the form

$$\begin{aligned}\Delta(\xi, \xi', \xi'') &\equiv \tfrac{1}{4}\mathrm{Tr}(\Pi(\xi)\Pi(\xi')\Pi(\xi''))\\ &= K(\xi, \xi') + K(\xi', \xi'') + K(-\xi, \xi'')\end{aligned} \tag{164}$$

Now, any Clifford algebra can be written

$$C(r, s) = C_0(r, s) \oplus C_1(r, s) \oplus C_2(r, s) \oplus \cdots \oplus C_n(r, s) \tag{165}$$

$$\equiv C_{\mathrm{even}}(r, s) \oplus C_{\mathrm{odd}}(r, s) \tag{166}$$

where $C_k(r, s)$ consists of all powers of k different generators, i.e., C_0 consists of the scalars, C_1 of the generators, C_2 of products of the form $\gamma_i\gamma_j$ and so on, while C_{even}, C_{odd} consist of all linear combinations of products with an even and odd number of generators, respectively. To each such Clifford algebra two Lie groups are defined, namely[11]

$$\mathrm{Pin}(r, s) = \langle C_1\rangle \tag{167}$$

$$\mathrm{Spin}(r, s) = \mathrm{Pin}(r, s) \cap C_{\mathrm{even}}(r, s) \tag{168}$$

and $\mathrm{Pin}(r, s)$ is homomorphic to $O(r, s)$. It furthermore turns out that the corresponding Lie algebra $spin(r, s)$ is isomorphic to $so(r, s)$, so we do not get any new classical phase spaces from that, even though the corresponding

[11] The symbol $\langle C_1\rangle$ denotes the group generated by all the unit vectors in C_1, i.e., the group of products of generators γ_i.

Lie groups $Spin(r, s)$ are inequivalent to any classical matrix group in all but a few cases (Göckeler and Schücker, 1987), as $Spin(r, s)$ is a covering group of $SO(r, s)$. If, on the other hand, we do *not* consider $spin(r, s)$ as a classical Lie algebra, but instead consider it as the Lie algebra of the nonclassical Lie group $Spin(r, s)$, which is built from the Clifford algebra $C(r, s)$, then we *can* get new phase spaces, namely Grassmann spaces. This leads, then, to an alternative for the classical Lie algebras $so(r, s)$, as we have already seen for $su_2 = so(3)$. By construction, we must also have morphisms between the two alternatives, the classical differentiable manifold $SO(r, s)/H$ and the Grassmann spaces, thus allowing for the translation of problems of analysis on $SO(r, s)/H$ into problems involving G-numbers, a possibility which should be of quite some practical importance. One important difference is that, considering $so_{r,s}$ as a Lie algebra, we get a symplectic manifold, whereas considering it as a Clifford algebra, we get an odd-dimensional Grassmann space.

8. QUANTUM-LIE ALGEBRAS, INTERMEDIATE STATISTICS, ETC.

We will make some very brief (and rather sketchy) comments on the possible extension of the above method to quantum groups and related structures.

Given a (semisimple) Lie algebra **g**, we can form its corresponding *quantum universal algebra* $U_q(\mathbf{g})$ (Fuchs, 1992), which is a deformed Lie algebra. A basis for this can be chosen in analogy with the ordinary Lie algebra case such that it satisfies

$$[H^i, H^j] = 0$$

$$[H^u, H^j_{\pm}] = \pm A^{ji} E^j_{\pm}$$

$$[E^i_+, E^j_-] = \delta^{ij} \lfloor H^i \rfloor$$

where the only new thing is the appearance of

$$\lfloor H^i \rfloor = \lfloor H^i \rfloor_q \equiv \frac{q^{H^i/2} - q^{-H^i/2}}{q^{1/2} - q^{-1/2}}$$

on the right-hand side above. It is here that the quantum deformation q enters. We see that we can carry the formalism developed above for an arbitrary Lie algebra **g** over to its quantum universal algebra $U_q(\mathbf{g})$ by making the substitution

$$H^i \to \lfloor H^i \rfloor$$

in the definition of $Q(u, v; u', v')$, but not in Π. The logarithm of Q would then be a highly nonlinear function of H^i (it will be linear in $\lfloor H^i \rfloor$, though) and this nonlinearity will be a measure of the deformation. The corresponding quantum fiber bundle will now involve a double deformation of a classical vector bundle. Would "second-quantized fiber bundle" be a good name for such a structure?

We will just make some very brief comments on some further generalizations. Bosons are described in terms of commutators and fermions in terms of anticommutators. Introducing the spin s of the underlying field (integral for bosons, half-integal for fermions), we can write this as

$$[a_k, a_l^\dagger]_s \equiv a_k a_l^\dagger - (-1)^{2s+1} a_l^\dagger a_k = \delta_{kl} \tag{169}$$

An obvious generalization is to allow s to be any rational or even real number; we can then define statistics interpolating between Bose–Einstein and Fermi–Dirac statistics. Now, given two fermionic operators a, $a^\dagger$, we can define bosonic ones by defining

$$A = \alpha a, \qquad A^\dagger = \beta a^\dagger$$

Requiring that (α, β) are G-numbers which anticommute with the Fermi operators, we have $[A, A^\dagger] = \alpha\beta$, so when $\beta = \alpha$ and α is normalized to unity, then A, $A^\dagger$ are ordinary Bose operators. We can do a similar trick here by formally defining "numbers" which satisfy

$$[\alpha, \beta]_s = 0 \Rightarrow \alpha\beta = (-1)^{2s+1} \beta\alpha$$

This will give us an ordinary Lie algebra in the formal operators A_k, $A_k^\dagger$ and we know the WWM formalism for these, hence we can extend it to these intermediate statistics as well by using this little trick. The symplectic product would then read

$$(\alpha, \beta) \wedge (\alpha', \beta') = \alpha\beta' - (-1)^{2s+1} \beta\alpha' \tag{170}$$

This leads to an alternative for quantum Lie algebras. If we have relations like

$$a_k a_l^\dagger = q R_{kl}{}^{k'l'} a_{l'}^\dagger a_{k'} \tag{171}$$

then we need coordinates satisfying

$$x_k y_l = q R_{kl}{}^{k'l'} y_{l'} x_{k'} \tag{172}$$

$$x_k x_l = x_l x_k \tag{173}$$

$$y_k y_l = y_l y_k \tag{174}$$

So Γ would become a *braided space* or a quantum space. We can thus establish morphisms between ordinary manifolds [**g** considered as a Lie algebra, or $U_q(\mathbf{g})$ considered as a deformation of **g**], Grassmann manifolds

[$\mathbf{g} = so(r, s)$ considered as a spin algebra], and braided spaces [$U_q(\mathbf{g})$ considered as an algebra of transformations on such spaces]. Such morphisms are of interest in their own right, as they show relationships between what would otherwise appear as unrelated areas of mathematics.

One could further consider general nonlinear algebras, i.e., algebraic structures satisfying

$$[\lambda_i, \lambda_j] = iF_{ij}(\lambda) \tag{175}$$

of which a quantum Lie algebra is but a particular case. As always, we will have different options for the classical phase space, depending upon how we interpret this algebraic structure [i.e., as a deformation of an ordinary (super-)Lie algebra, or as an algebra of automorphisms of some noncommutative structure á la braided spaces]. One could study parafermions and parabosons in this way, for instance.

9. COMMENT ON FINITE GROUPS

All our emphasis so far has been on "continuous" structures, Lie algebras, and structures derived therefrom; before we move on to discuss operator algebras, it is therefore appropriate to make a few comments on finite groups. Given a finite group G, we can construct its algebra $C(G)$; this is the set of formal linear combinations $\Sigma_{i=1}^{|G|} \alpha_i g_i$ with $\alpha_i \in \mathsf{F}$ and $G = \{g_i | i = 1, \ldots, n = |G|\}$. The coefficients $\alpha_i = \alpha(g_i)$ are thus functions $G \rightarrow \mathsf{F}$, and we can assume G is a topological group with α_i continous, which explains the reason for the terminology $C(G)$.[12]

The idea is again, of course, to use the following:

Definition 7. Let $G = \{e, g_1, \ldots, g_{n-1}\}$ be a finite group; we define

$$\Pi(u) = \exp\left(i \sum_{j=1}^{n'} u_j g_j - i \sum_{j=n'+1}^{n-1} \lambda_j(u) g_j \right) \tag{176}$$

where $n = |G|$, $n' = |G| - |H|$, with H denoting the largest Abelian subgroup of the commutator $G' = \{g_1 g_2 g_1^{-1} g_2^{-1} | g_1, g_2 \in G\}$, and where we have chosen the numbering such that $g_0 = e \in H$ is the neutral element.

This function Π is considered as a formal power series, and the coefficients u_j, $\lambda(u_j)$ can in general be noncommutative (they are just formal quantities). In the case where we have an identification of G with a group of transformations over some finite field (or division ring or even just principal ideal domain), such as the *Chevalley groups* $A_k(\mathsf{F})$, $B_k(\mathsf{F})$, $C_k(\mathsf{F})$, $D_k(\mathsf{F})$ which

[12] The natural topology is the discrete one, of course, making all sets open and all functions continuous.

generalize the usual Lie algebras of the same names, (Carter, 1977), it would be natural to let u_j, λ_j belong to this finite field (or division ring) F.

Thus there is an ambiguity in the definition for finite groups, as we have no *a priori* candidate for F, the field (or even just ring) to which the coefficients in the algebra $C(G)$ of G belongs. Choosing an infinite field like $\mathsf{F} = \mathsf{R}$ or $\mathsf{F} = \mathsf{C}$ would just give us ordinary Lie algebras, whereas an infinite field such as Q, $\mathsf{Q}(\alpha_1, \ldots, \alpha_n)$, with α_i transcendent over Q, would lead to something slightly different, of use, perhaps in Galois theory, while choosing a finite field $\mathsf{F} = \mathsf{Z}_p = \mathsf{Z}/p\mathsf{Z}$, p a prime, or $\mathsf{F} = GF(p^n)$ (the so-called *Galois field*) would lead to something very different, namely a finite, discrete set (i.e., a kind of lattice) as the classical phase space.

Let us furthermore notice that for finite groups we have $g^n = e$ for any element g of the group, and so the exponential is well defined, and can in fact be "decomposed" as

$$\Pi(u) = 1 + \sum_{j=0}^{n-1} \pi_j(u) g_j \tag{177}$$

For the cases su_2, su_3 and Clifford algebras we have a similar decomposition, which was very useful for practical calculations. The functions $\pi_j(u)$ are Taylor series if the field has characteristic zero, and polynomials otherwise.

Before we look at some examples, let us notice that the phase space of a Galois extension $\mathsf{F}(\alpha)$ can be obtained from that of the original field F in a simple manner. Let $\mathsf{F}(\alpha)$ have dimension n as a vector space over F, i.e., $|\mathsf{F}(\alpha) : \mathsf{F}| = n$, then $\mathsf{F}(\alpha) = \mathsf{F} \oplus \alpha\mathsf{F} \oplus \cdots \oplus \alpha^{n-1}\mathsf{F}$, so any element in the Galois extension can be written as $u = u_0 + u_1\alpha + \cdots + u_{n-1}\alpha^{n-1}$. So the transition $\mathsf{F} \to \mathsf{F}(\alpha)$ can be written $u \mapsto u(\alpha) = u_0 + u_1\alpha + \cdots + u_{n-1}\alpha^{n-1}$. We have thus proven the following result:

Proposition 13. Let F be any field and let α be transcendent over F; for any Chevalley algebra $\mathbf{g}$ over F we then have

$$\Gamma_{\mathbf{g}}(\mathsf{F}(\alpha)) = \Gamma_{\mathbf{g}}(\mathsf{F}) \otimes \mathsf{F}(\alpha) \tag{178}$$

This result is very similar to the ones for loop algebras or complexifications we saw earlier.

9.1. Examples of Finite Groups

To develop the formalism I will just give two examples, the permutation group S_3 and the Chevalley group $A_1(\mathsf{F})$, F some field (finite or infinite).

For the permutation groups S_3, A_3 we have the multiplication table as shown in Table II, with A_3 being the subgroup made up by $\{e, g_1, g_4\}$, which is also the largest Abelian subgroup. From this we get

Table II. The Multiplication Table of $G = S_3$

	e	g_1	g_2	g_3	g_4	g_5
e	e	g_1	g_2	g_3	g_4	g_5
g_1	g_1	g_4	g_3	g_5	e	g_2
g_2	g_2	g_5	e	g_4	g_3	g_1
g_3	g_3	g_2	g_1	e	g_5	g_4
g_4	g_4	e	g_5	g_2	g_1	g_3
g_5	g_5	g_3	g_4	g_1	g_2	e

$$\Pi(u) = \exp(-i\lambda_1 g_1 + iu_2 g_2 + iu_3 g_3 - i\lambda_2 g_4 + iu_5 g_5) \tag{179}$$

$$= 1 + \pi_0(u)e + \sum_{i=1}^{5} \pi_i(u) g_i \tag{180}$$

where

$$\pi_i(u) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \alpha_i^{(n)}, \qquad i = 0, 1, \ldots, 5 \tag{181}$$

with the coefficients $\alpha_i^{(n)}$ given by the recursion relations

$$\begin{aligned}
\alpha_0^{(n+1)} &= \alpha_0^{(n)}\alpha_0^{(1)} + \alpha_1^{(n)}\alpha_4^{(1)} + \alpha_2^{(n)}\alpha_2^{(1)} + \alpha_3^{(n)}\alpha_3^{(1)} + \alpha_4^{(n)}\alpha_1^{(1)} + \alpha_5^{(n)}\alpha_5^{(1)} \\
\alpha_1^{(n+1)} &= \alpha_0^{(n)}\alpha_1^{(1)} + \alpha_1^{(n)}\alpha_0^{(1)} + \alpha_2^{(n)}\alpha_5^{(1)} + \alpha_3^{(n)}\alpha_2^{(1)} + \alpha_4^{(n)}\alpha_4^{(1)} + \alpha_5^{(n)}\alpha_3^{(1)} \\
\alpha_2^{(n+1)} &= \alpha_0^{(n)}\alpha_2^{(1)} + \alpha_1^{(n)}\alpha_5^{(1)} + \alpha_2^{(n)}\alpha_0^{(1)} + \alpha_3^{(n)}\alpha_1^{(1)} + \alpha_4^{(n)}\alpha_3^{(1)} + \alpha_5^{(n)}\alpha_4^{(1)} \\
\alpha_3^{(n+1)} &= \alpha_0^{(n)}\alpha_3^{(1)} + \alpha_1^{(n)}\alpha_2^{(1)} + \alpha_2^{(n)}\alpha_4^{(1)} + \alpha_3^{(n)}\alpha_0^{(1)} + \alpha_4^{(n)}\alpha_5^{(1)} + \alpha_5^{(n)}\alpha_1^{(1)} \\
\alpha_4^{(n+1)} &= \alpha_0^{(n)}\alpha_4^{(1)} + \alpha_1^{(n)}\alpha_1^{(1)} + \alpha_2^{(n)}\alpha_3^{(1)} + \alpha_3^{(n)}\alpha_5^{(1)} + \alpha_4^{(n)}\alpha_0^{(1)} + \alpha_5^{(n)}\alpha_2^{(1)} \\
\alpha_5^{(n+1)} &= \alpha_0^{(n)}\alpha_5^{(1)} + \alpha_1^{(n)}\alpha_3^{(1)} + \alpha_2^{(n)}\alpha_1^{(1)} + \alpha_3^{(n)}\alpha_4^{(1)} + \alpha_4^{(n)}\alpha_2^{(1)} + \alpha_5^{(n)}\alpha_0^{(1)}
\end{aligned}$$

subject to

$$\alpha_0^{(1)} = 0, \qquad \alpha_i^{(1)} = u_i \quad \text{for } i = 2, 3, 5, \qquad \alpha_1^{(1)} = -\lambda_1, \qquad \alpha_4^{(1)} = -\lambda_2 \tag{182}$$

The dimensionality of the "phase space" (with a field of characteristic zero as underlying field) is then $|S_3| - |A_3| = 6 - 3 = 3$. The deformed addition is rather complicated, namely

$$\begin{pmatrix} u_2 \\ u_3 \\ u_5 \end{pmatrix} \oplus \begin{pmatrix} u_2' \\ u_3' \\ u_5' \end{pmatrix} = \begin{pmatrix} u_2 + u_2' - u_3\lambda_1' - \lambda_2 u_3' - u_5\lambda_2' - \lambda_1 u_5' \\ u_3 + u_3' - \lambda_2 u_2' - u_2\lambda_2' - \lambda_2 u_5' - u_5\lambda_1' \\ u_5 + u_5' - \lambda_1 u_3' - u_2\lambda_1' - u_3\lambda_2' - \lambda_2 u_2' \end{pmatrix} \tag{183}$$

For finite-dimensional Lie algebras, the "undeformed," or "zeroth-order" antisymmetric two-form ω_0 is the coefficient, to the lowest order, of the Cartan elements, hence (for a general Lie algebra, with root decomposition as in the text)

$$\omega_0(u, v, u', v') = \sum_{\alpha} (u_\alpha v'_\alpha - u'_\alpha v_\alpha) \tag{184}$$

and this is then the analogue of the Poisson bracket when dim Γ is even. In our case of a finite group, the analogous quantity is

$$\begin{aligned} \omega_0(u, u') &= u_2 u'_5 - u'_2 u_5 + u_3 u'_2 - u'_3 u_2 + u_5 u'_3 + u_5 u'_3 \\ &= \begin{vmatrix} -1 & u_2 & u'_2 \\ -1 & u_3 & u'_3 \\ -1 & u_5 & u'_5 \end{vmatrix} \end{aligned} \tag{185}$$

The Chevalley group of $A_1(\mathsf{F})$ over any field (finite or infinite) F is defined from the relations

$$[e, f] = h, \qquad [h, e] = e, \qquad [h, f] = -f \tag{186}$$

Letting $A_1(\mathsf{Z})$ denote the Z-linear span of these elements, we get a Lie algebra; for any field F we then put

$$A_1(\mathsf{F}) \equiv A_1(\mathsf{Z}) \otimes \mathsf{F} \tag{187}$$

For $\mathsf{F} = \mathsf{R}$ we get $sl_2(\mathsf{R}) = so_3 = su_2$, whereas for $\mathsf{F} = \mathsf{C}$ we get their respective complexifications. For a finite field $\mathsf{F} = GF(p^n)$ [with $GF(p) = \mathsf{Z}_p$] we get something completely new, and for $\mathsf{F} = \mathsf{Q}$ we get $sl_2(\mathsf{Q})$. Let us concentrate upon $\mathsf{F} = \mathsf{Z}_p$ for now. The phase space cannot simply, as for the infinite field R, C, be diffeomorphic to $\{x, y, z \in \mathsf{F} | x^2 + y^2 + z^2 = 1\}$, as spheres of different radii will contain an unequal number of points in the discrete case.

The subgroup H is just the diagonal subgroup, and hence is isomorphic to $\mathsf{F}^\times$, where $\mathsf{F}^\times$ denotes the set of invertible elements in F (for F a field and not just a division ring, this is $\mathsf{F}\backslash\{0\}$). Hence, since the group with Lie algebra $A_1(\mathsf{F})$ is $PSL_2(\mathsf{F})$ (Carter, 1977)

$$\Gamma_{A_1}(\mathsf{F}) \simeq PSL_2(\mathsf{F})/\mathsf{F}^\times \tag{188}$$

For an infinite field such as Q or one of its Galois extensions, this is a "manifold" of dimension 2, as for $\mathsf{F} = \mathsf{R}$, C, whereas for finite fields it is a finite set of points. For $\mathsf{F} = GF(p^n)$ for some prime p and some integer n, we have

$$|\Gamma| = \frac{1}{(2, p^n - 1)} p^{2n}(p^{2n} - 1) - (p^n - 1) \tag{189}$$

where we have used $|GF(p^n)| = p^n$ and where (a, b) denotes the greatest common divisor of a, b. In the special case $n = 1$, in which case $GF(p) \simeq \mathbb{Z}_p$, we thus get a set consisting of 11 points for $p = 2$, 34 for $p = 3$, and so on.

I will leave the discussion of finite groups at this point to give a summary of properties derived so far, and then go onto operator algebras. The further development of a WWM formalism for finite groups will certainly be of interest in its own right (applications to pure algebra, Galois theory, and algebraic geometry spring to mind), but I do not know of any physical situation which could serve as a motivation.

10. SUMMARY OF PROPERTIES

I will finish this discussion with a summary of the algebraic properties of the WWM formalism I have been developing. The formalism consists basically of (1) Π and Q, the maps defining the Weyl transformation and its algebraic properties, and (2) the set $C(\Gamma)$ of functions $\Gamma \to \mathbf{C}$, where Γ is the classical phase space. The basic correspondence is

$$A_W(\xi) \equiv \text{Tr}\, \Pi(\xi)\hat{A}$$

$$\hat{A} \equiv \int_\Gamma \Pi(\xi) A_W(\xi)\, d\mu$$

where the Weyl transform $\hat{A} \mapsto A_W$ is an isomorphism $U(\mathbf{g}) \to C(\Gamma)$. The operator-valued function Π can be viewed as a "translation" operator and satisfies (in a local coordinate patch)

$$\Pi(\xi)\Pi(\xi') = \Pi(\xi \oplus \xi')Q(\xi \times \xi')$$

The operations $\oplus$ and $\times$ were referred to as the deformed addition and symplectic product, respectively. For an Abelian algebra $\xi \oplus \xi' = \xi + \xi'$, and thus the deformation is a measure of the noncommutativity. Furthermore, the classical phase space Γ is a vector space if the algebra is Abelian and a symplectic manifold if $\mathbf{g}$ is semisimple or obtained from a semisimple Lie algebra by a central extension or by adding an Abelian algebra. Its dimensionality is

$$\dim \Gamma = \dim \mathbf{g} - \text{rank}\, \mathbf{g} \equiv n - 1$$

and for $n = \dim \mathbf{g} < \infty$ we have

$$\Gamma_{\mathbf{g}} = G/H$$

where G is the smallest connected Lie group having $\mathbf{g}$ as its Lie algebra, while H is similar, but for the Cartan subalgebra of $\mathbf{g}$.

We discovered some very nice properties of (Π, Q, Γ), namely

$$\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2 \Rightarrow \Pi_{\mathbf{g}} = \Pi_{\mathbf{g}_1}\Pi_{\mathbf{g}_2} \quad \text{and} \quad Q_{\mathbf{g}} = Q_{\mathbf{g}_1}Q_{\mathbf{g}_2}$$

$$\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2 \Rightarrow \Pi_{\mathbf{g}} = \Pi_{\mathbf{g}_1}\Pi_{\mathbf{g}_2} \quad \text{and} \quad Q_{\mathbf{g}} = Q_{\mathbf{g}_1}Q_{\mathbf{g}_2}q_Z \quad \text{if} \quad [\mathbf{g}_1, \mathbf{g}_2] \subseteq Z(\mathbf{g})$$

$$\mathbf{h} \text{ ideal in } \mathbf{g} \Rightarrow \Pi_{\mathbf{g}} = \Pi_{\mathbf{g}/\mathbf{h}}\Pi_{\mathbf{h}} \quad \text{and} \quad Q_{\mathbf{g}} = Q_{\mathbf{g}/\mathbf{h}}Q_{\mathbf{h}}$$

which allows us to study central extension very easily (for instance, to express the WWM formalism for an affine Kac–Moody algebra in terms of the WWM formalism for a loop algebra). Another very important property was

$$\Gamma(\mathbf{g} \otimes C^\infty(M)) = \Gamma(C^\infty(M \to \mathbf{g})) \simeq C^\infty(M \to \Gamma(\mathbf{g}))$$

which allows us to gauge an algebra and extend out WWM formalism easily, in particular we can go to the loop algebra $M = S^1$. A similar result holds for Galois extensions of the base field $\mathsf{F} \to \mathsf{F}(\alpha_1, \ldots, \alpha_n)$

$$\Gamma_{\mathbf{g}}(\mathsf{F}(\alpha_1, \ldots, \alpha_n)) \simeq \Gamma_{\mathbf{g}} \otimes \mathsf{F}(\alpha_1, \ldots, \alpha_n)$$

For $\mathsf{F} = \mathsf{R}$, $\alpha = \pm i$ we get a result about complexifications.

A final result relates to morphisms $\phi: \mathbf{g}_1 \to \mathbf{g}_2$, i.e., structure-preserving maps between algebras (homomorphisms for Lie algebras; Jordan maps, i.e., linear maps preserving the anticommutator, for fermions; super-Lie homomorphisms for super-Lie algebras; and so on). Any such morphism induces a map $\Phi: C(\Gamma_1) \to C(\Gamma_2)$, where Γ_i is the phase space of $\mathbf{g}_i$. Consider the commutative diagram

$$\begin{array}{ccc} U(\mathbf{g}_1) & \stackrel{\phi}{\to} & U(\mathbf{g}_2) \\ \Pi_1 \big\downarrow & & \big\downarrow \Pi_2 \\ C(\Gamma_1) & \stackrel{\Phi}{\to} & C(\Gamma_2) \end{array}$$

Using that Π_1 is an isomorphism, we can define

$$\Phi = \Pi_2 \circ \phi \circ \Pi_1^{-1}$$

and then Φ is well defined and unique.

We can use this to carry topological and algebraic structure from $\mathbf{g}$ through $U(\mathbf{g})$ to $C(\Gamma)$. Suppose, for instance, that $\mathbf{g}$ is a normed or seminormed space, i.e., it is endowed with a map $\rho: \mathbf{g} \to \mathsf{R}$ which is sublinear $[\rho(A +$

$B) \leq \rho(A) + \rho(B)]$ and positive homogeneous $[\rho(\alpha A) = |\alpha|\rho(A)$ with α a scalar]. Noting that $\Gamma(\mathbf{R}) = \{0\}$, i.e., $C(\Gamma(\mathbf{R})) \simeq \mathbf{R}$ (similar for $\mathbf{C}$, of course), we have the commutative diagram

$$\begin{array}{ccc} \mathbf{g} & \xrightarrow{\rho} & \mathbf{R} \\ \downarrow & & \downarrow \\ C(\Gamma) & \xrightarrow{\tilde{\rho}} & C(\Gamma(\mathbf{R})) \simeq \mathbf{R} \end{array}$$

Thus $C(\Gamma)$ is a normed or seminormed space whenever $\mathbf{g}$ is. Hence $C(\Gamma)$ is a Banach space if and only if $\mathbf{g}$ is, and the mapping Π becomes an isometry in this case. Similarly, if $\mathbf{g}$ comes equipped with an inner product, i.e., a sesquilinear map $\mathbf{g} \times \mathbf{g} \to \mathbf{C}$, then Π induces a sesquilinear form on $C(\Gamma)$, which then becomes Hilbert if and only if $\mathbf{g}$ is a Hilbert space. The diagram is

$$\begin{array}{ccccc} \mathbf{g} & \to & \mathbf{g} \times \mathbf{g} & \to & \mathbf{C} \\ \downarrow & & \downarrow & & \downarrow \\ C(\Gamma) & \to & C(\Gamma) \times C(\Gamma) & \to & \mathbf{C} \end{array}$$

We should note that semisimple Lie algebras come with a natural nondegenerate bilinear form and will thus give pre-Hilbert spaces.

Let us also note that this shows that our construction is in fact independent of the representation: considering $\mathbf{g}_1, \mathbf{g}_2$ to be two faithful irreducible representations of a given Lie algebra $\mathbf{g}$, i.e., we have isomorphisms ρ_i: $\mathbf{g} \to \mathbf{g}_i \subseteq gl_{n_i}$, this induces an isomorphism $\mathbf{g}_1 \to \mathbf{g}_2$ and hence their two classical phase spaces will be equivalent. The diagram is

$$\begin{array}{ccccc} U(\mathbf{g}_1) & & \xrightarrow{\rho_2 \circ \rho_1^{-1}} & & U(\mathbf{g}_2) \\ & \nwarrow^{\rho_1} & & \nearrow^{\rho_2} & \\ \Pi_1 \downarrow & & U(\mathbf{g}) & & \downarrow \Pi_2 \\ & & \downarrow \Pi & & \\ & & C(\Gamma) & & \\ & \swarrow_{\tilde{\rho}_1} & & \searrow_{\tilde{\rho}_2} & \\ C(\Gamma_1) & & \xrightarrow{\widetilde{\rho_2 \circ \rho_1^{-1}}} & & C(\Gamma_2) \end{array}$$

with

$$\widetilde{\rho_2 \circ \rho_1^{-1}} = \tilde{\rho}_2 \circ \tilde{\rho}_1^{-1} \tag{190}$$

Furthermore, any diffeomorphism α: $\Gamma_1 \to \Gamma_2$ induces a map α_*: $C(\Gamma_1) \to C(\Gamma_2)$, which then leads to a map $\tilde{\alpha}$: $U(\mathbf{g}_1) \to U(\mathbf{g}_2)$, which allows us to study the group of maps α of one manifold onto another in a new, more algebraic way.

We have the following result:

Propositon 14. If $\mathbf{g}$ is a normed algebra, then so is $C(\Gamma)$; if $\mathbf{g}$ has an inner product, then so does $C(\Gamma)$. Thus if $\mathbf{g}$ is Hilbert or Banach, then so is $C(\Gamma)$.

All of the above hold for a very large class of algebraic structures as we have seen.

11. *C**-ALGEBRAS

It would be interesting to go on to an even larger class of algebras such as C*-algebras. The general idea is to construct an isomorphism

$$\mathcal{A} \to C(\Gamma)$$

between a C*-algebra and an algebra of functions on some manifold Γ. For *Abelian* algebras such an isomorphism is already known [the Gel'fand theorem; see, e.g., Bratteli and Robinson (1979) and Murphy (1990)]

$$\mathcal{A} \simeq C_0(X)$$

where C_0 denotes the functions vanishing at infinity and X is some locally compact Hausdorff space (the spectrum or maximal ideal space of A) which is compact if and only if $\mathcal{A}$ contains the identity (Bratteli and Robinson, 1979; Murphy, 1990). Our WWM formalism would then provide us with a *non-Abelian Gel'fand theorem*. One should note that the basic ingredient in Gel'fand's theorem is the concept of a *character* on an Abelian C*-algebra, i.e., a linear map χ: $\mathcal{A} \to \mathbf{C}$ such that $\chi(AB) = \chi(A)\chi(B)$; X is the space of such maps, and is hence a subset of the dual $\mathcal{A}^*$ of $\mathcal{A}$. The WWM formalism gives a natural generalization of this: $\chi(A) = A_W$; the product rule then reads $\chi(AB) = \chi(A) * \chi(B)$ and we could refer to the Weyl transform as a *generalized character*. The major problem is the construction of Γ (the Abelian case uses $\mathcal{A} \subset \mathcal{A}^{**}$ and $X \subset \mathcal{A}^*$, hence $\mathcal{A}$ can be viewed as a function on X; it then relies on the Stone–Weierstrass theorem to prove the isomorphism, and this is difficult to generalize to non-Abelian algebras).

Any non-Abelian C^*-algebra is isomorphic to a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on some separable Hilbert space $\mathcal{H}$. The method developed in the previous sections can thus be seen as a special case, namely the case of finite-dimensional C^*-algebras, and we now want to go further. A particular important subalgebra $\mathcal{B}$ is $\mathcal{K} = \mathcal{B}_0(\mathcal{H})$ of *compact operators*, i.e., the operators for which the image of the unit ball $\{x \in \mathcal{H} \mid \|x\|^2 \leq 1\}$ is compact. The elements of this subalgebra can be approximated by finite matrices; in fact (Murphy, 1990; Wegge–Olsen, 1993)

$$\mathcal{K} = \lim_{\rightarrow} gl_n(\mathbf{C})$$

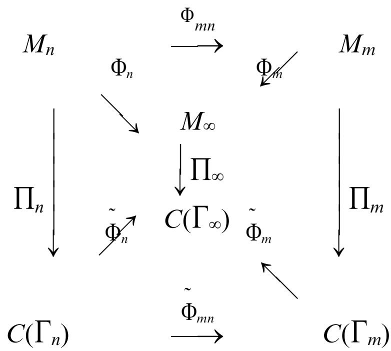
where the $\lim_{\rightarrow}$ is understood as the *inductive limit*, hence $\mathcal{K}$ is the completion (in norm-topology) of $gl_\infty(\mathbf{C})$. This suggest that the case of compact operators is the next simplest case to treat.[13] And in fact we can use the very definition of inductive limit to construct directly the corresponding classical phase space. Recall that the inductive limit requires a *directed system* $\{A_i, \Phi_{ij}\}_{i\in\mathcal{I}}$, i.e., a family of objects A_i indexed by an upward filtering index set $\mathcal{I}$ (i.e., a set $\mathcal{I}$ such that whenever $i, j \in \mathcal{I}$, a $k \in \mathcal{I}$ exists such that $k > i$ and $k > j$) and with a morphism Φ_{ij}: $A_j \rightarrow A_i$ whenever $j > i$. The inductive limit A_∞ is then the object $\cup_{\mathcal{I}} A_i$ with morphisms Φ_i: $A_i \rightarrow A_\infty$ such that

$$\begin{array}{ccc} A_j & \xrightarrow{\Phi_j} & A_\infty \\ {\scriptstyle \Phi_{ij}}\big\downarrow & \nearrow{\scriptstyle \Phi_i} & \\ A_j & & \end{array}$$

commutes.

Denoting by Π_n the WWM map from $M_n = gl_n(\mathbf{C})$ into $C(\Gamma_n)$, where Γ_n is the classical phase space corresponding to M_n, we get the following diagram:

[13] A C^*-algebra which can be obtained as the inductive limit of matrix algebras is known as an *AF-algebra*, an “approximately finite dimensional” algebra. Thus our methods can be generalized to these.



Expressed in formulas, we have

$$C(\Gamma(\mathcal{K})) = C(\Gamma_\infty)) = \lim_{\rightarrow} C(\Gamma_n) \tag{191}$$

The map Π_∞ is given by

$$\Pi_\infty(A) \equiv \lim_{n\to\infty} \Pi_n(P_n A P_n) \tag{192}$$

where Π_n is, as in the diagram, the Weyl map for gl_n and where P_n is the projection $\mathcal{K} \to gl_n$; these constitute an approximate unit for $\mathcal{K}$ (i.e., $P_n A \to A \; \forall A \in \mathcal{K}$) and the above construction is then well defined.

If we could extend our scheme to $\mathcal{B}(\mathcal{H})$, then we would be able to treat any C^*-algebra; thus our next problem is to find out how to go from $\mathcal{K} = \mathcal{B}_0$ to $\mathcal{B}$. One way is to write down an exact sequence[14]

$$0 \to \mathcal{K} \to \mathcal{B} \to \mathcal{B}/\mathcal{K} \to 0$$

where $\mathcal{B}/\mathcal{K}$ is known as the *Calkin algebra*; this shows that $\mathcal{B}$ is an *extension* of the algebra $\mathcal{K}$ by the Calkin algebra. There is another way of obtaining $\mathcal{B}$ from $\mathcal{K}$, namely by the use of what is known as the *multiplier algebra* $\mathcal{M}(A)$ of a C^*-algebra; this is defined as the largest utilization of A,[15] and can be constructed as follows. Suppose A acts nondegenerately on some Hilbert space $\mathcal{H}_1$ (this is always possible to arrange); then $A \subseteq \mathcal{B}(\mathcal{H}_1)$, and we put

$$\mathcal{M}(A) = \{x \in \mathcal{B}(\mathcal{H}_1) | xA \subseteq A \wedge Ax \subseteq A\} \tag{193}$$

[14] A sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is said to be *exact* if the kernel of β is the image of α, i.e., going twice ($\beta \circ \alpha$) gives zero, and this is the only way of getting zero. Hence $0 \to A \xrightarrow{\alpha} B$ is exact if and only if α is injective, and $A \xrightarrow{\alpha} B \to 0$ is exact if and only if α is surjective. This notion is easily generalized to longer sequences; we simply demand the kernel of one map to be equal to the image of the previous one.

[15] I.e. the largest algebra constructed from A containing A itself and a unit element 1.

Equivalently, $\mathcal{M}(A)$ is the completion in the topology induced by the seminorms $x \mapsto \|xa\|$ and $x \mapsto \|ax\|$, where $x \in \mathcal{B}(\mathcal{H}_1)$ and $a \in A$ (this topology is known as the *strict topology*). The basic vector is

$$\mathcal{M}(\mathcal{K}) = \mathcal{B} \tag{194}$$

Thus if we can find a way of extending the WWM formalism for a given C^*-algebra A consisting of compact operators ($A \subseteq \mathcal{K}$) to its multiplier algebra $\mathcal{M}(A)$, then we will have extended our WWM formalism to all C^*-algebras. Another interesting possibility, closely related to this, is the study of the WWM formalism for arbitrary extensions of A. This would also be an interesting exercise in the case of Lie algebras, as would the study of Inönü–Wigner contractions.

Before doing this, let us look at the simplest (smallest) unitization A^+ of A; when A is not itself unital, then $A^+ \simeq A + 1\mathbf{C}$, i.e., $x = a + \lambda$, $x \in A^+$, $a \in A$, $\lambda \in \mathbf{C}$ with a natural product $(a + \lambda)(b + \mu) = ab + \lambda b + \mu a + \lambda\mu$. Any morphism ϕ: $A \to B$ between C^*-algebras induces a morphism ϕ^+: $A^+ \to B^+$ given by

$$\phi^+(a + \lambda) \equiv \phi(a) + \lambda$$

Letting $B = C(\Gamma)$ and $\phi = \Pi$, we get

$$C(\Gamma^+) = C(\Gamma(A^+)) \simeq C(\Gamma) \times \mathbf{C} \tag{195}$$

Any function in $C(\Gamma^+)$ is thus a pair $(f(x), \lambda)$, where f: $A \to \mathbf{C}$ and $\lambda \in \mathbf{C}$. This implies that $\Gamma(A^+) \equiv \Gamma^+$ is constructed by the adjoining of a point to $\Gamma(A) = \Gamma$; the scalar λ is then the value assigned to f at this extra point, i.e., we can consider Γ^+ to be the one-point compactification of Γ; in standard symbols

$$\Gamma^+ = \alpha\Gamma \tag{196}$$

For C^*-algebras the adjoining of a unit does not lead to the old phase space plus some isolated point, as we always have sequences $e_n \to 1$, $e_n \in A$ (approximate units); so the new phase space, which is again the old one with some point added, must be just as connected as the original one, thus leading to a compactification as argued above. For Lie algebras we do not have any sequences corresponding to approximate units, and hence get isolated points.

Now, the Gel'fand theory for Abelian C^*-algebras gives exactly this relationship, too, which seems to imply that our scheme is indeed in some sense the noncommutative version of Gel'fand's. Similarly, we can see that any unitization of A leads to a compactification of Γ:

$$\text{unitization of } A \to \text{compactification of } \Gamma$$

Let A_1, A_2 be two different unitizations of A; then $A_1 \subseteq A_2$ implies $\Gamma_1 \subseteq \Gamma_2$, where $\Gamma_i = \Gamma(A_i)$. Now, the smallest unitization should thus correspond to the smallest compactification (which we also saw that it did) and the largest unitization, the multiplier algebra $\mathcal{M}(A)$, to the largest compactification $\beta\Gamma$, the Stone–Čech compactification. Thus

$$\Gamma_{\mathcal{M}(A)} = \beta\Gamma_A \tag{197}$$

and the *corona algebra* $\mathcal{M}(A)/A$ becomes isomorphic to $C(\beta\Gamma)/C(\Gamma) \simeq C(\beta\Gamma\backslash\Gamma)$. We thus have the following result:

Proposition 15. Let A be a C^*-algebra and let $A^+ = A + 1\mathbf{C}$ denote the smallest possible unitization and $\mathcal{M}(A)$ the multiplier algebra. Suppose the classical phase space of A is Γ; then

$$\Gamma(A^+) \simeq \alpha\Gamma \quad \text{(one-point compactification)}$$
$$\Gamma(\mathcal{M}(A)) \simeq \beta\Gamma \quad \text{(Stone–Čech compactification)}$$

We are now through; $\mathcal{M}(\mathcal{K}) = \mathcal{B}$, and, as we mentioned, any non-Abelian C^*-algebra sits as a subalgebra inside $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

With the relationship between unitizations and compactification clarified, we can go on to extensions. We say that B is an *extension* of A by C if

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

is exact.

Now any morphism $0 \to A \to B$ induces a unique morphism $B \to \mathcal{M}(A)$; in fact we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & A & \overset{\alpha}{\to} & B & \overset{\beta}{\to} & C & \to 0 \\
 & \Vert & & \downarrow \sigma & & \downarrow \tau & \\
0 \to & A & \to & \mathcal{M}(A) & \to & \mathcal{M}(A)/A & \to 0
\end{array}$$

The morphism τ is known as the *Busby invariant*; it characterizes the extension and is unique (Wegge-Olsen, 1993). We suppose that we know the classical phase spaces of A and C and we want to find it for the larger algebra $B \subseteq A \oplus C$. It turns out (Wegge-Olsen, 1993) that B can be constructed from τ and A in the following way:

$$B \simeq \{a \oplus c \in \mathcal{M}(A) \oplus C \mid \pi(a) = \tau(c)\} \tag{198}$$

where π is the canonical quotient map $\mathcal{M}(A) \to \mathcal{M}(A)/A$. We say that B is the *pullback* of $\mathcal{M}(A)/A$ along π and τ. This implies that $C(\Gamma_B)$ is a kind of "diagonal" subspace of $C(\beta\Gamma_A) \oplus C(\Gamma_C)$, namely:

Proposition 16. If A, B, C are C^*-algebras and if B is an extension of A by C, then

$$C(\Gamma_B) \simeq \{f \in g \in C(\beta\Gamma_A) \oplus C(\Gamma_C) \,|\, \tilde{\pi}(f) = \tilde{\tau}(g) \in C(\beta\Gamma_A \backslash \Gamma_A)\} \quad (199)$$

In this way we are able to construct the classical phase space of an extension from its Busby invariant τ and the classical phase spaces of the other algebras. We see, e.g., that $C(\Gamma_A)$ has codimension one when C is an Abelian C^*-algebra.

Admittedly, the WWM formalism put forward in this paper is rather formal as far as C^*-algebras are concerned; we were only able to show how in principle one could construct classical phase spaces, and we saw that Γ_∞, the classical phase space of the algebra of compact operators, could be expressed as a direct limit of $\Gamma_n = \Gamma(gl_n)$. We have not given explicit constructions for other C^*-algebras, though. The next natural step will be to study specific C^*-algebras, e.g., the *irrational rotation algebras* A_θ, which correspond closely to the Heisenberg algebra, the *Toeplitz algebra* (generated by the shift operator), which can be seen as a kind of limit of so_l or su_l, its generalization the so-called *Cuntz algebras*, and so on. This will be sketched in the next section.

12. EXAMPLES OF C^*-ALGEBRAS

We will begin with algebras generated by shift operators. First of all, we will consider the Hilbert space $l^2(\mathbb{Z})$, i.e., the space of all square-summable sequences of complex numbers with the set of integers as their index set. An important operator on this space is the *bilateral shift*

$$S|n\rangle = |n + 1\rangle \quad (200)$$

where $\{|n\rangle\}$, $n \in \mathbb{Z}$, denotes an orthonormal basis. The adjoint operator S^* similarly satisfies

$$S^*|n\rangle = |n - 1\rangle \quad (201)$$

and we see that S is unitary. We can form the C^*-algebra $A = C^*(S)$ generated by S (and thus also including S^*). Clearly, A is Abelian and hence isomorphic to $\mathbb{C}[[X, X]]$, i.e., $\Gamma = \mathbb{C}$. A much more interesting case comes about when we consider not the integers, but only the natural numbers $\mathbb{N}$ as index set. We then get the *unilateral shift*, which is only an isometry: $S^*S = 1$, but $SS^* \neq 1$; in fact, $SS^* = (1 - \delta_{n1}) = 1 - P_1$, where P_1 is the projection unto $|1\rangle$, i.e.,

$$[S, S^*] = P_1 \quad (202)$$

The corresponding C^*-algebra is known as the *Toeplitz algebra* and will be denoted by $\mathcal{T}$. This algebra is one of the best studied and important C^*-algebras. It can also be seen as an extension of $\mathcal{K}$, the compact operators, by $C(S^1)$, the Abelian C^*-algebra of continuous functions on the circle. Any element in $\mathcal{T}$ can be written as $x = \sum_{n,m=0}^{\infty} x_{nm} S^n (S^*)^m = \sum_{n,m} x_{nm} T_{nm}$, where $T_{nm} = S^n S^{*m}$. The commutator of these generators is easily seen to be

$$\begin{aligned}[T_{nm}, T_{n'm'}] = {} & \theta(n' - m)T_{n+n'-m,m'} + \theta(m - n')T_{n,m-n'+m'} \\ & - \theta(n - m')T_{n+n'-m',m} - \theta(m' - n)T_{n',m+m'-n} \\ & + \delta_{n'm}T_{nm'} - \delta_{nm'}T_{n'm} \end{aligned} \tag{203}$$

We note that $\{T_{n0}\}$, $\{T_{0n}\}$ form two (isomorphic) Abelian subalgebras. Any element of the classical phase space will then be of the form

$$\xi(x, y) = \sum_{nm} \xi_{nm} x^n y^m \tag{204}$$

with

$$(\xi_{mn})^\dagger = \overline{\xi_{nm}} \tag{205}$$

Hence $\Gamma_{\mathcal{T}}$ consists of analytical functions $S^1 \times S^1 \to \mathbf{C}$. The "translation operator" Π has the form

$$\Pi_{\mathcal{T}}(\xi) = \exp\left[i \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \xi_{nm} T_{nm} + i \sum_{m=0}^{\infty} \lambda_m(\xi) T_{m0} \right] \tag{206}$$

The only *a priori* restriction on the coefficients ξ_{nm} is that $\xi \in l_1(\mathbf{N}_0 \times \mathbf{N})$, the set of absolute summable series indexed by $\mathbf{N}_0 \times \mathbf{N}$ with $\mathbf{N}_0 = \{0, 1, 2, 3, \ldots\}$. This can also be interpreted as functions in $H^1(S^1 \times S^1)$, the *Hardy space* of absolute integrable functions $f(x, y)$ such that f vanishes whenever $x, y < 0$, and we finally end up with

$$\begin{aligned}\Gamma_{\mathcal{T}A} &= H^1(S^1 \times S^1)/H^1(S^1) \simeq \{f \in H^1(S^1 \times S^1) \mid f|_{\text{diag}} = 0\} \\ &\equiv \tilde{H}^1(S^1 \times S^1)\end{aligned} \tag{207}$$

The Toeplitz algebra can also be defined in another way, namely as the C^*-algebra generated by operators of the form $x \mapsto T_\phi x = P(\phi x)$, where $\phi \in C(S^1)$ and P is the projection $L^2(S^1) \to H^2(S^1)$, so it is not surprising that the Hardy spaces H^p turn up. We get H^1 and not H^2, as we only have a norm and not a sesquilinear form on our operator algebra (if we could define a "Hilbert–Schmidt" subalgebra, then it would be isomorphic to $\tilde{H}^2$), and we get the space $S^1 \times S^1$ and not just S^1 because we have to take S and S^* as independent quantities, thus giving rise to an underlying two-dimensional space.

The Toeplitz algebra is not Abelian, so it is not surprising that we get an infinite-dimensional phase space, which we can then represent as a space of functions. The elements in the Toeplitz algebra get represented by nonlinear functionals in this manner. The next obvious step is the so-called *Cuntz algebra* $\mathbb{O}_n$ spanned by n isometries S_i subject to

$$\sum_{i=1}^{n} S_i S_i^* = 1 \tag{208}$$

i.e., their range projections $S_i S_i^*$ cover the entire space. By analogy with the Toeplitz case we get

$$\Gamma_{\mathbb{O}_n} = \tilde{H}^1(\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{2n})$$

The next important case is A_θ, the *rotation algebras*, where $\theta \in \mathbb{R}$; these are generated by two unitaries u, v subject to

$$uv = e^{i2\pi\theta} vu \tag{210}$$

Let $T_{nm} = u^n v^m$; we quickly arrive at the algebra

$$\begin{aligned}[T_{mn}, T_{m'n'}] = {} & (\delta_{nm'} e^{-in2\pi\theta} - \delta_{n'm} e^{-in'2\pi\theta}) T_{m+m',n+n'} \\ & + \theta(n - m') e^{-im'2\pi\theta} T_{m,n+n'-m'} + \theta(m' - n) e^{-in2\pi\theta} T_{m+m'-n,n'} \\ & - \theta(n' - m) e^{-im2\pi\theta} T_{m',n+n'-m} - \theta(m - n') e^{-in'2\pi\theta} T_{m+m'-n',n}\end{aligned} \tag{211}$$

Here $\theta(n)$ is the Heaviside step function. We see that when θ is a rational number, we can choose n, m, n', m' in a nontrivial way and still get a vanishing commutator (e.g., $n = m'$, $n' = m$, and $n - m$ an even number), whereas for θ irrational this is not possible. Thus for $\theta \in \mathbb{Q}$ we can have either a larger maximal Abelian subalgebra or we can embed $l^1(\mathbb{Z})$ in more than two (inequivalent) ways. When the angle θ is irrational we get

$$\Gamma_{A\theta} = l^1(\mathbb{Z}^2)/l^1(\mathbb{Z}) = \{(\xi_{nm} \in l^1(\mathbb{Z}^2) \mid x_{nn} = 0\} \equiv \tilde{l}^1(\mathbb{Z}^2) \tag{212}$$

represented as a space of sequences, or equivalently as a space of functions

$$\Gamma_{A\theta} = \tilde{L}^1(S^1 \times S^1) \equiv \{f \in L^1(S^1 \times S^1) \mid f|_{\text{diag}} = 0\} \tag{213}$$

Further examples can of course be thought of, but we will stop for now. The spaces we found are listed in Table III. The reason why we always had Γ of the form $\mathcal{F}(\Gamma_0)$ where $\mathcal{F}$ denotes some class of functions with Γ_0 compact (indeed of the form $S^1 \times S^1 \times \cdots \times S^1$) was that we always had a finite number of generators.

Table III. The Classical Phase Spaces Γ for a Number of C^*-Algebras

Space	C^*-algebra
$\mathbb{C}[[X, X]]$	Bilateral shift
$\tilde{H}^1(S^1 \times S^1)$	Unilateral shift/Toeplitz algebra
$\tilde{H}(\underbrace{S^1 \times \cdots \times S^1}_{2n})$	Cuntz algebra $\mathbb{O}_n$
$\tilde{L}^2(S^1 \times S^1)$	Irrational rotation algebra

13. OUTLOOK: TOWARD A GENERAL DEQUANTIZATION AND QUANTIZATION PROCEDURE

The method we have been developing in the previous sections constitutes a general "dequantization" mechanism: to a given quantum phase space we associate a classical phase space and we identify the quantum operators with functions on this space. So far this formalism has been developed for Lie, super-Lie, and quantum-Lie algebras as well as C^*-algebras.

If we want to include noncontinuous functions, we would have to go to von Neumann algebras instead, and this would be the natural next step. Let me just sketch what one should probably do. A *weight* on a von Neumann algebra $\mathcal{A}$ (Sunder, 1987) is a linear map ω: $\mathcal{A}_+ \to \mathbb{R}_+ \cup \{\infty\} = [0,\infty]$; we call it a *trace* if $\omega(A^* A) = \omega(AA^*)$.[16] Any von Neumann algebra possesses a trace which is semifinite (i.e., the subset of $\mathcal{A}$ given by $\omega(|A|) < \infty$ is dense in some specific topology). This should be the mapping that replaces the usual trace, and we could define

$$\mathcal{A}^p(\omega) \equiv \{A \in \mathcal{A} \,|\, \omega(|A|^p) < \infty\} \tag{214}$$

We then want a map Π such that

$$\Pi: \quad \mathcal{A}^p(\omega) \to L^p(\Gamma, d\mu_\omega)$$

is an isomorphism. Continuing as before, we would write

$$A_W(\xi) = \omega(\Pi(\xi)A)$$

$$A = \int \Pi(\xi) A_W(\xi)\, d\mu_\omega$$

assuming that we can still use the same $\Pi(\xi)$ in both directions. The mapping $A \leftrightarrow A_W$ is then also denoted by Π as before.

[16] We mentioned the possibility of this more abstract definition already in the section on Lie algebras, but this is the first time we really do need it. For finite-dimensional algebras any trace as defined above is just the usual matrix trace (up to a constant). A further generalization, suited for K-theoretic analysis, is to replace the trace by an arbitrary cyclic cocycle.

The elements of $\mathscr{A}$ which do not belong to any of the subspaces $\mathscr{A}^p$ would then, by extension of Π, be mapped into measurable, but not absolutely integrable functions (i.e., in none of the L^p-spaces), i.e.,

$$\Pi: \quad \mathscr{A} \to M(\Gamma, d\mu_\omega)$$

where $M(\Gamma, d\mu)$ denotes the set of measurable functions on Γ. We can extend Π to all of $\mathscr{A}$ by using its semifiniteness, and assuming Π to be continuous in some given topology. We know, formally at least, that we *can* extend our WWM formalism to von Neumann algebras as well, as these are, by definition, subalgebras of $\mathscr{B}(H)$ for some Hilbert space H, i.e., they lie inside some C^*-algebra. Similarly, given any C^*-algebra A, we can use the GNS construction to obtain an isomorphism π of A onto a subalgebra of $\mathscr{B}(H)$ for some (in general huge) Hilbert space H; the algebra $B = \pi(A)''$ will then be a von Neumann algebra containing A, where A'' denotes the double commutant of an algebra [i.e., set of all elements which commutes with any element of $\mathscr{B}(H)$ commuting with all of A (Sunder, 1987; Murphy, 1990)].

As far as operator algebras are concerned, one might also consider "regularizing" the trace, by replacing it by some cyclic cocycle cohomologous to it.

Another important development would be the inverse of what we have been doing so far, namely constructing a general *quantization* mechanism, which, given a symplectic manifold, deforms it and yields a non-Abelian algebra of functions which is isomorphic to an operator algebra. Symbolically:

$$\{\cdot, \cdot\}_{\mathrm{PB}} \to [\cdot, \cdot]_{\mathrm{M}} \to [\cdot, \cdot]$$

This would allow us to quantize arbitrary classical theories. Some progress has been made over the past decades in this direction; it is, for instance, known that any symplectic manifold admits a twisted product (Flato and Sternheimer, 1980). In this case we should probably make much more use of the symmetries of the classical phase space, finding some way, this restricts the corresponding quantum phase spaces algebraic structure.

An interesting application of this formalism would be to index theorems; as the WWM formalism establishes a link between operators and functions, and thus between algebra, geometry, and topology, it ought to be useful in this context. It also opens up the possibility of characterizing the topology of certain manifolds by purely algebraic means, and, on the other hand, to give geometrical/topological interpretations of otherwise purely algebraic concepts. What could turn out to be particularly useful is the various possible choices of phase spaces for the algebras $so(r, s)$, depending on whether one looks upon them as Lie or Clifford algebras, or, indeed, as deformed algebras, establishing connections between ordinary manifolds, Grassmann spaces, and braided spaces, respectively. Especially for harmonic and/or functional analysis on these spaces, this relationship could very well prove very powerful.

As a final comment, one should notice that WWM quantization might help resolve problems of operator ordering (each WWM map defined its own unique operator ordering prescription) and renormalization. The usual problems with renormalizability stems from the multiplication of distributions, and this is ill defined for ordinary products, but might be quite reasonable for twisted products, or by "regularizing" by replacing the trace by a cyclic cocycle cohomologous to it.

14. CONCLUSION

We have seen how we can generalize the Wigner–Weyl–Moyal formalism first to the case where the quantum phase space is an arbitrary Lie algebra of finite or infinite dimension. We also saw how to relate the WWM formalism for a loop algebra $\mathbf{g}_{\text{loop}}$ or a Kac-Moody algebra $\hat{\mathbf{g}}_k$ based on some ordinary, finite-dimensional, semisimple Lie algebra $\mathbf{g}$ to the WWM formalism of $\mathbf{g}$ itself. We were furthermore able to treat fermionic degrees of freedom, i.e., anticommutators, and hence to include super-Lie algebras as well. Next, it was indicated how deformed Lie algebras, quantum Lie algebras, could be treated, too, and how the WWM formalism of a q-deformed Lie algebra $\mathbf{g}_q$ could be related to that of the original algebra. Some comments were also made on intermediate statistics. As our standard example we took su_2, and we saw how the corresponding classical phase space turned out to be S^2. Naively, the classical phase space corresponding to a Lie algebra of rank l and dimension n is R^{n-l}, but we realized that the noncommutativity of the algebra resulted in a deformation of this vector space, so in the end, the classical phase space became only locally isomorphic to R^{n-l}, i.e., became an $(n - l)$-dimensional real manifold. The curvature of this manifold was a measure of the noncommutativity of the Lie algebra. The algebra structure induced an addition and a symplectic product on the classical phase space, which were deformations of the corresponding operations in the flat space. We should emphasize that although we have only used Lie algebras over the field of complex numbers, essentially the same analysis should be possible to carry out with any basefield, e.g., finite fields, thus giving us *Chevalley algebras*, or even just division rings (the quaternions, for instance). Some simplifications do occur in our case, though, as C is algebraically closed.

Carried over into the realm of C^*-algebras, the WWM formalism provides us with a kind of noncommutative Gelfand theorem, which differs from the usual Gelfand theorem in the Abelian case, though. We also speculated about how to extend the scheme to include von Neumann algebras. For reasons of space, we did not discuss the properties of the corresponding Wigner functions; this has to be left for future research.

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